Affine Deodhar Diagrams and Rational Dyck Paths

CANADAM 2025

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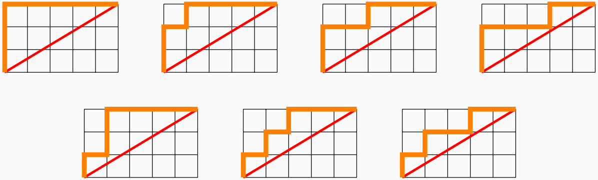
UC Los Angeles

Rational Catalan Numbers

For any $0 < k < n$ with $\gcd(k, n) = 1$, we define the **rational Catalan number** $C_{k,n-k}$ as the number of Dyck paths in a $k \times (n-k)$ rectangle.



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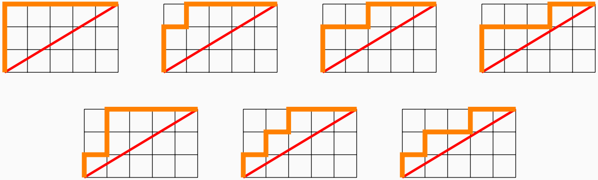

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For any $0 < k < n$ with $\gcd(k, n) = 1$,

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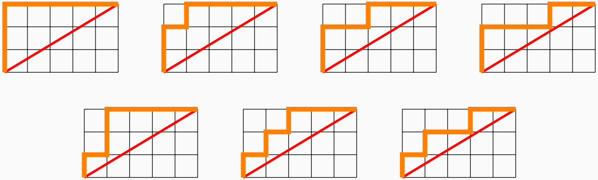
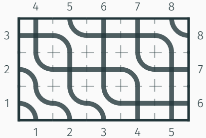


Proposition
For any $0 < k < n$ with $\gcd(k, n) = 1$,

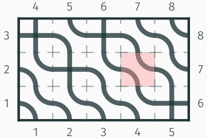
$$C_{k,n-k} = \frac{1}{n}\binom{n}{k}.$$

When $n = 2k + 1$, we recover the classical Catalan numbers.

A (k, n)-Deodhar Diagram (**Deogram**) is a filling of boxes of a $k \times (n-k)$ rectangle with crossings, , and elbows, , with



Example



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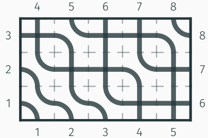

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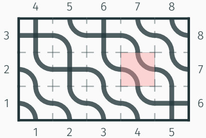

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(k, n)-Deograms

A (k, n)-Deodhar Diagram (**Deogram**) is a filling of boxes of a $k \times (n-k)$ rectangle with crossings, , and elbows, , with

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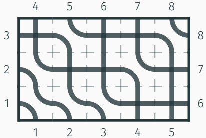

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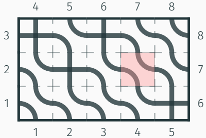

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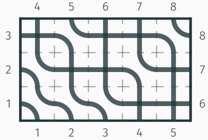

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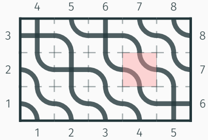

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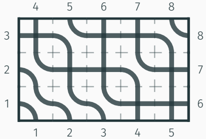

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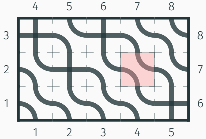
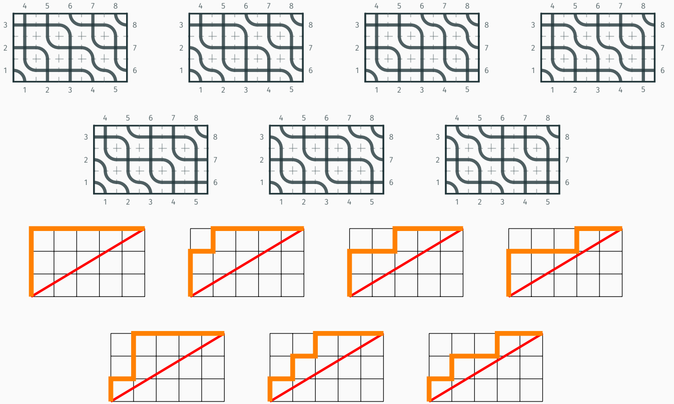


Non-example

Let $\mathrm{Deo}_{k,n}$ denote the set of (k,n)-Deograms.

Overview

Theorem (Galashin-Lam, '21)
For $0 < k < n$ with $\gcd(k, n) = 1$, $\mathrm{Deo}_{k,n}$ and $\mathrm{Dyck}_{k,n}$ are equinumerous.



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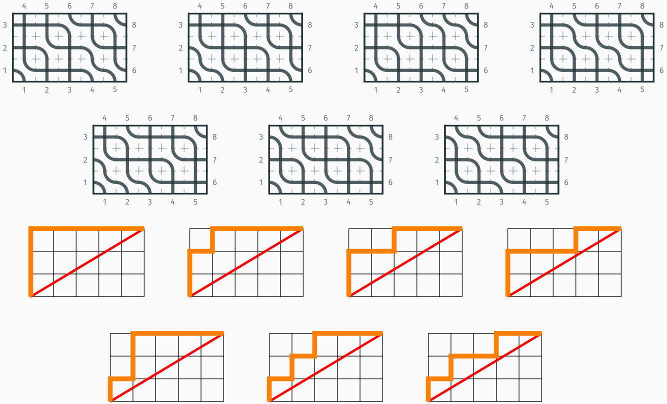

Theorem (M., '25+)
For $0 < k < n$ with $\gcd(k, n) = 1$, we find a bijection $\mathrm{Deo}_{k,n} \to \mathrm{Dyck}_{k,n}$.

Recurrence Motivation

Catalan Recurrence
For $n > 0$,

$$C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}.$$

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Proposition
For any $0 < k < n$ with $\gcd(k, n) = 1$,

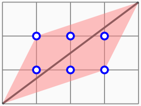
$$C_{k,n-k} = \frac{1}{n}\binom{n}{k}.$$

While we have a formula, we generally do not have a recurrence relation.

Convex Catalan Numbers

Convex Sets

Let Γ be a collection of lattice points inside a $k \times (n-k)$ rectangle. We call Γ **convex** if it contains every lattice point of its convex hull with the diagonal.



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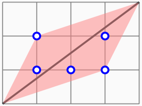

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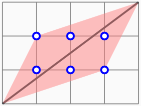

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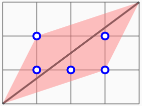

Non-example

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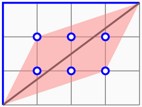
For every convex Γ, define $C_\Gamma = \#\,\mathsf{Dyck}(\Gamma)$, the number of lattice paths **strictly avoiding** Γ.



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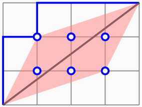

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Convex Recurrence

Lemma/Observation
For every convex Γ, there exists a pair of points $p_\Gamma, r_\Gamma \notin \Gamma$ such that $\Gamma' := \Gamma \cup \{p_\Gamma, r_\Gamma\}$ is convex.

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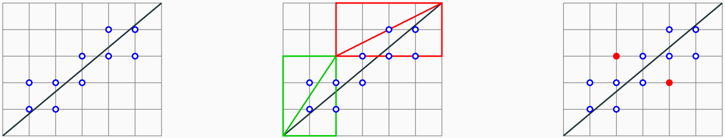
This gives us a simple recurrence relation.



$$\mathrm{Dyck}(\Gamma) \quad = \quad \mathrm{Dyck}(\Gamma_1) \cdot \mathrm{Dyck}(\Gamma_2) \quad + \quad \mathrm{Dyck}(\Gamma')$$

Permutations to Bounded Affine Permutations

But we need a tool to help us communicate between Deograms and Dyck paths.

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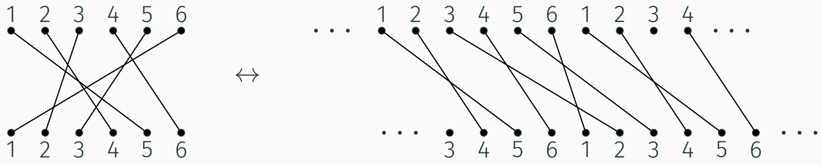
For $\bar{f} \in S_n$, we can associate a **bounded affine permutation** $f : \mathbb{Z} \to \mathbb{Z}$ to $\bar{f}$.



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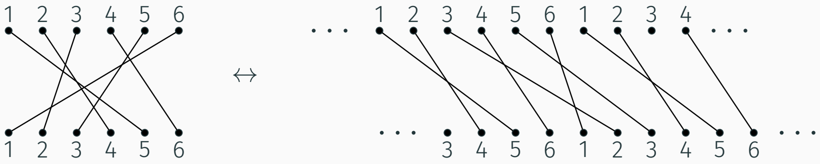
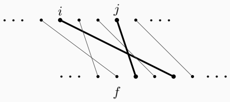
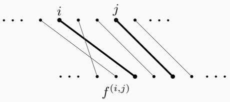


We can find k, the height of the rectangle as $\frac{1}{n}\sum_{i=1}^{n}(f(i) - i) = k$.

Positroid Catalan Numbers

Resolving crossings.





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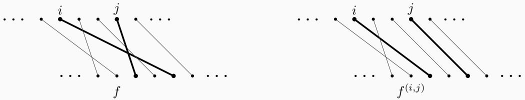

Inversion Multiset
We associate a lattice point for each inversion of f. The multiset $\Gamma(f)$ contains a point $\gamma(f_1^{(i,j)}) = (k, n-k)$ for each inversion $(i,j), i < j$, where f_1 is the cycle with i after resolving.

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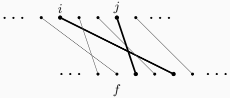

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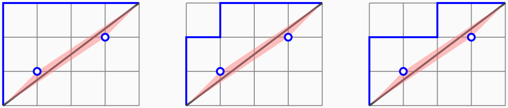

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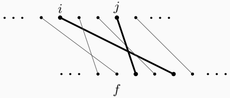

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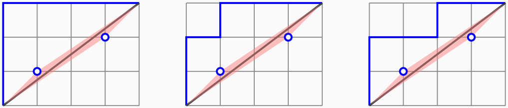
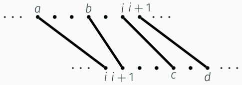
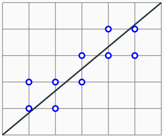


For $f_{k,n}(i) = i + k$, $\Gamma(f) = \emptyset$, so $C_{f_{k,n}} = \#\,\mathrm{Dyck}_{k,n-k} = C_{k,n-k}$.

Dyck Path Recurrence



f



$\mathrm{Dyck}(\Gamma(f)) \quad =$

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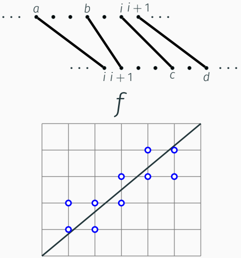

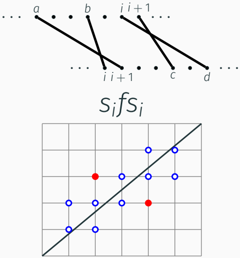

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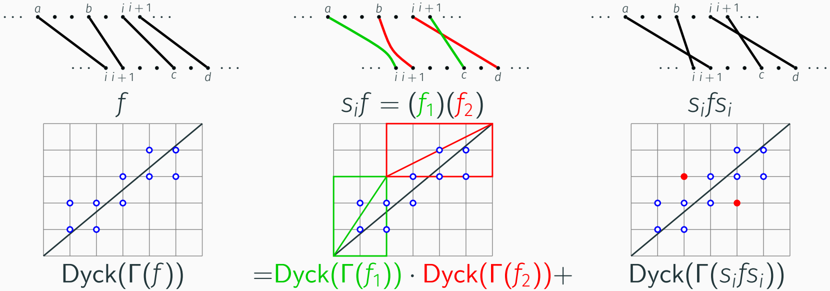

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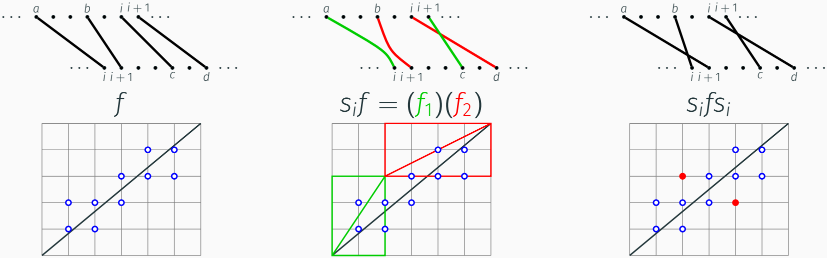
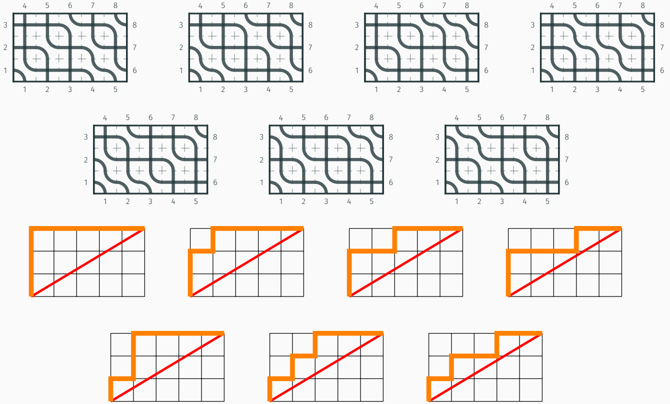


$$\mathrm{Dyck}(\Gamma(f)) = \mathrm{Dyck}(\Gamma(f_1)) \cdot \mathrm{Dyck}(\Gamma(f_2)) + \mathrm{Dyck}(\Gamma(s_i f s_i))$$

The positroid Catalan numbers, C_f,

1. recover the rational Catalan numbers, and
2. have a recurrence relation.

Recurrence Motivation



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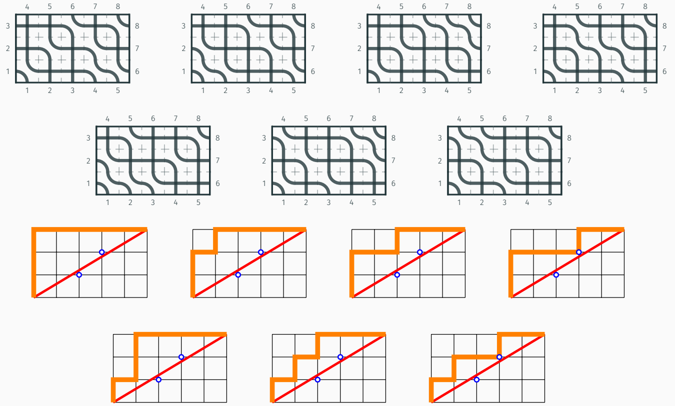

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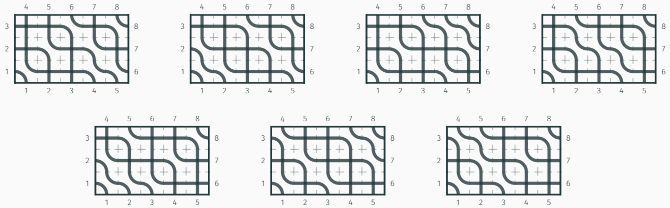

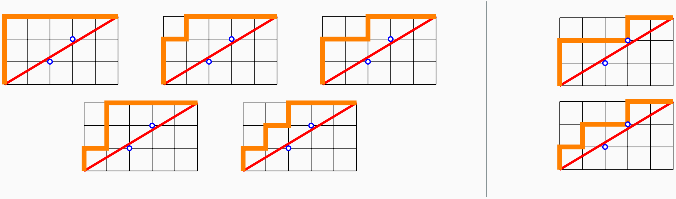

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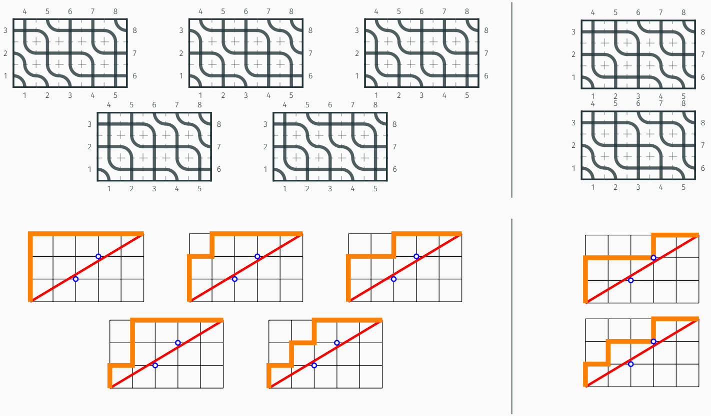

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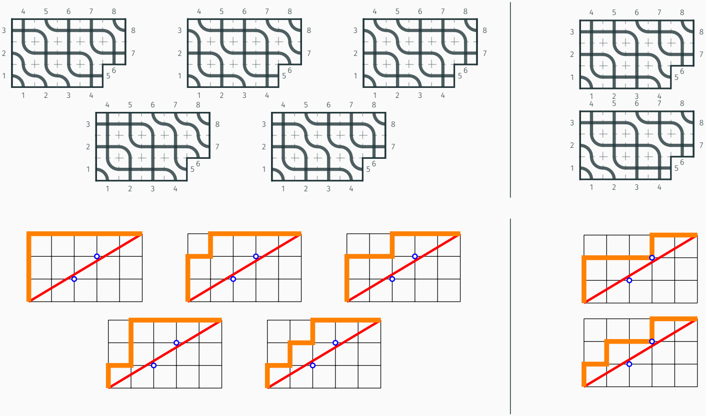

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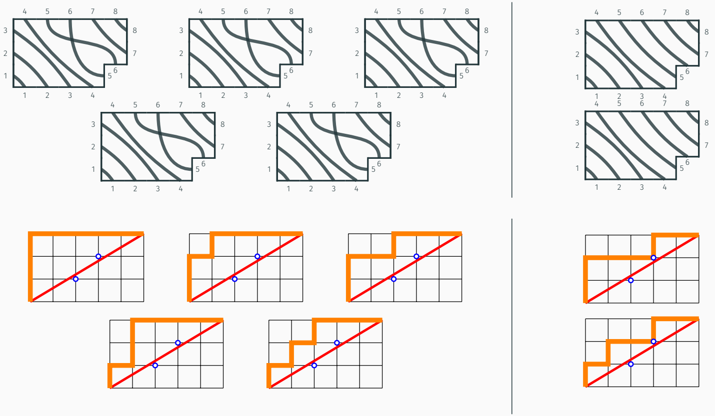
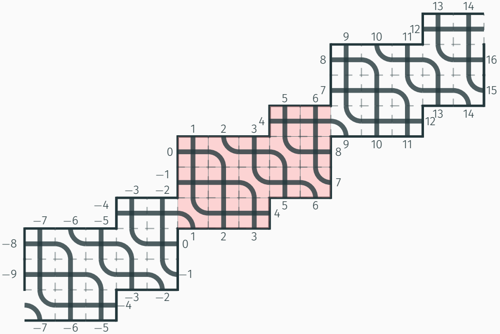


Definition (M., 25+)
A (maximal) f-**affine Deogram** is a *periodic filling* of the space between a path P with k up-steps and $n-k$ right steps and its vertical translate with:



Main Tool: Affine Deograms

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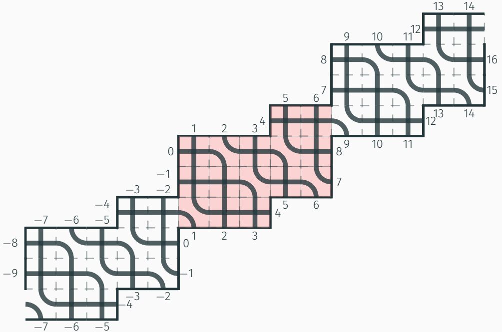

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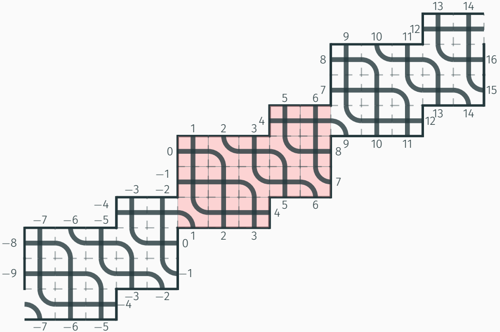

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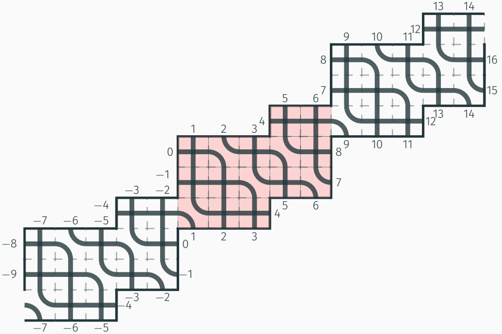

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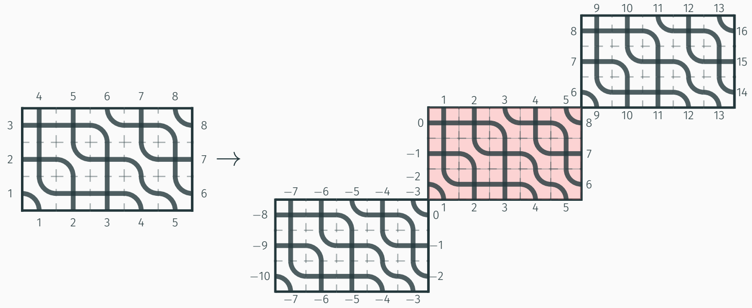

Moves on Affine Deograms

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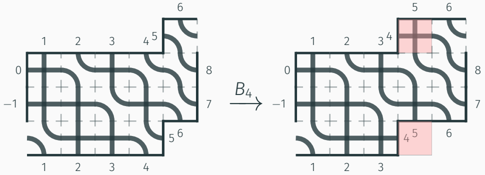

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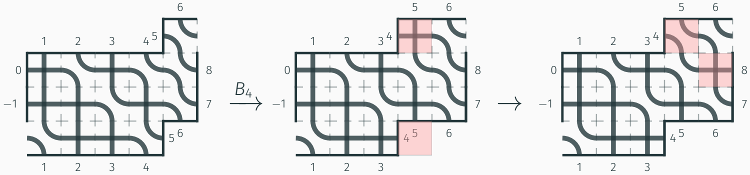

Box Addition/Removal

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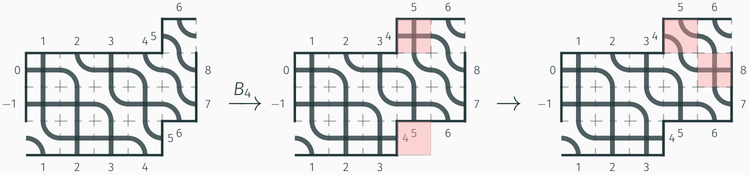


The move B_0 is why we need affine Deograms. It has no simple “lift” to rectangular Deograms.

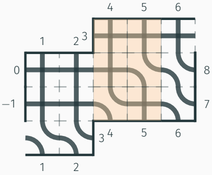
Moves on Affine Deograms

We have 3 moves on f-affine Deograms:

1. Box Addition/Removal
2. Zipper
3. Decoupling

Zipper

Motto: We cross wires below and locally apply Yang-Baxter moves until the crossing moves to the top of the path.



Zipper

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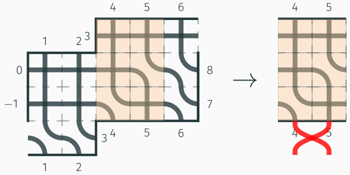

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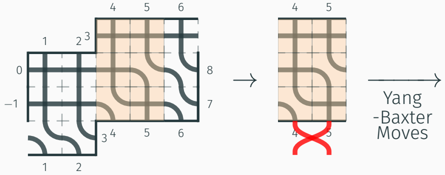

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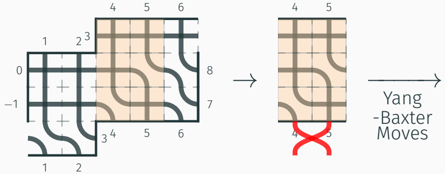

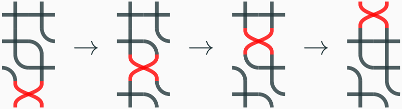

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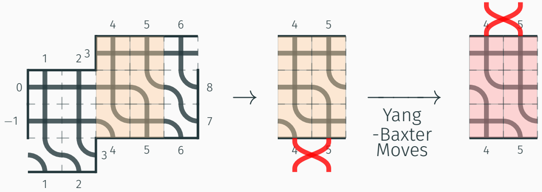

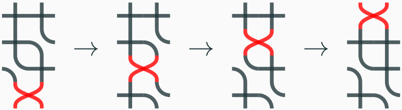

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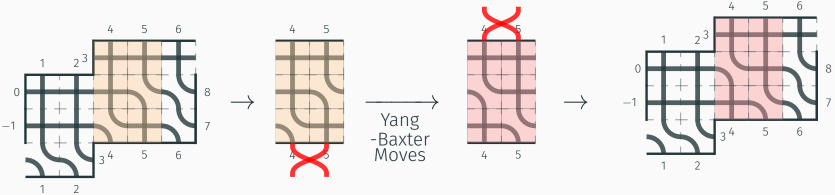

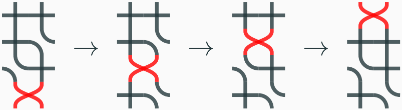

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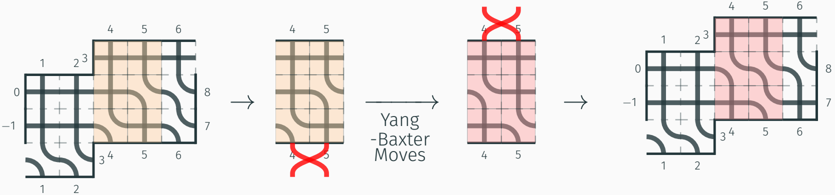

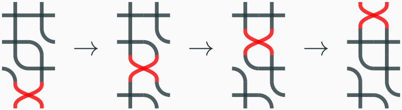

Moves on Affine Deograms

We have 3 moves on f-affine Deograms:

1. Box Addition/Removal
2. Zipper
3. **Decoupling**

Decoupling

Let $f = f_1 f_2 \dots f_r$ be a decomposition of $f \in \mathsf{B}_{k,n}$ into cycles. Then,

$$\#\,\mathsf{AffDeo}^{\max}_{f,P} = \prod_{i=1}^{r} \#\,\mathsf{AffDeo}^{\max}_{f_i,P_i}.$$

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We color the wires according to which cycle they are in. We then restrict ourselves to boxes with the same color.

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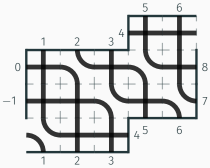

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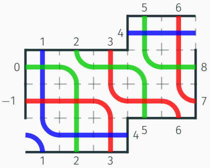

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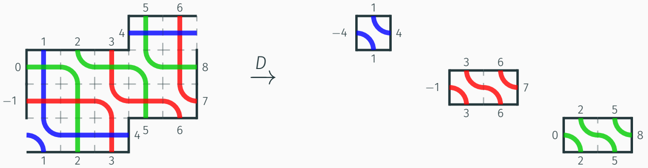
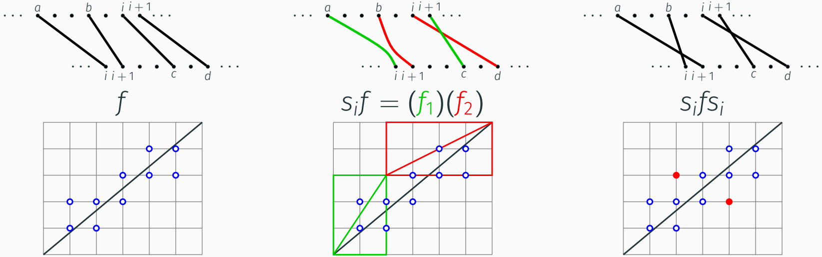


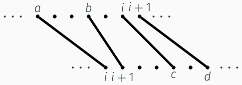
Okay, what do we have so far?



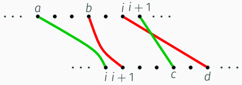
$$\mathrm{Dyck}(\Gamma(f)) = \mathrm{Dyck}(\Gamma(f_1)) \cdot \mathrm{Dyck}(\Gamma(f_2)) + \mathrm{Dyck}(\Gamma(s_i f s_i))$$

Affine Deogram Recurrence

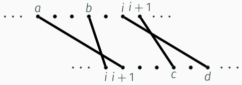
We get the same recurrence for affine Deograms.



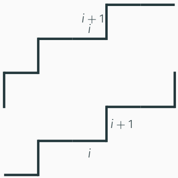
f



$s_i f = (f_1)(f_2)$



$s_i f s_i$



$$\mathrm{AffDeo}^{\max}_{f,P} = \mathrm{AffDeo}^{\max}_{f_1, s_i P} \cdot \mathrm{AffDeo}^{\max}_{f_2, s_i P} + \mathrm{AffDeo}^{\max}_{s_i f s_i, s_i P}$$

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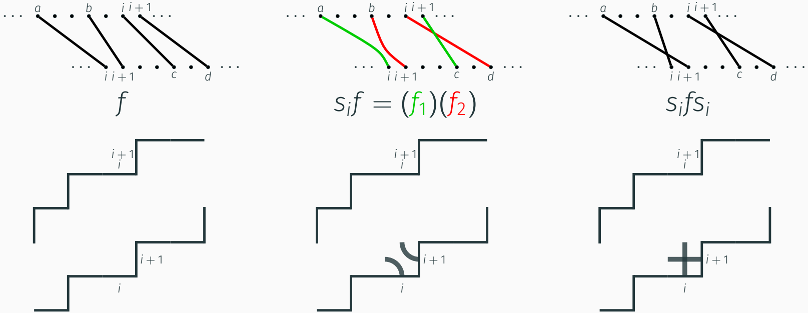

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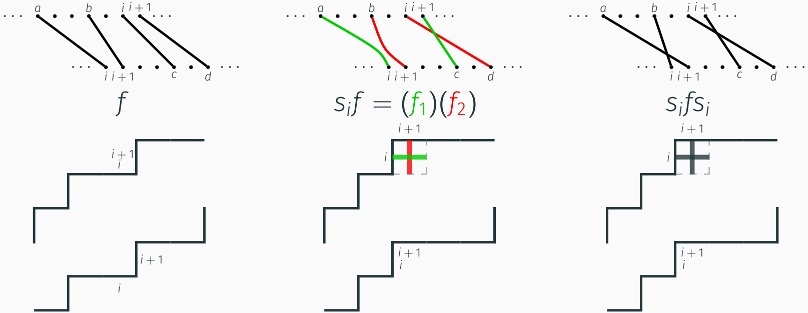

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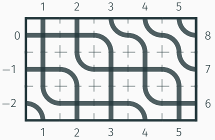
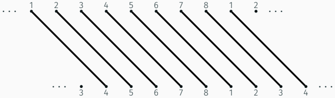
Full Recurrence Example





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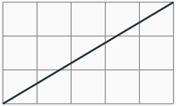
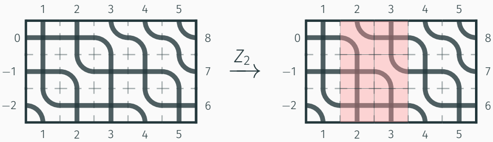
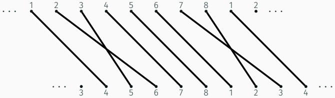


$\Gamma(f)$

Full Recurrence Example





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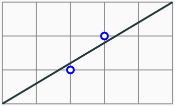

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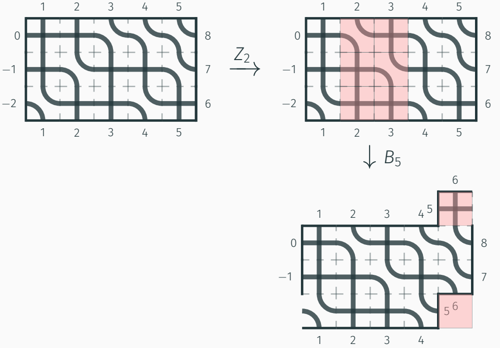

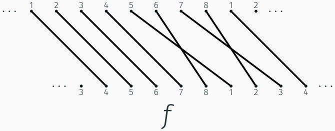

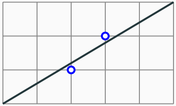

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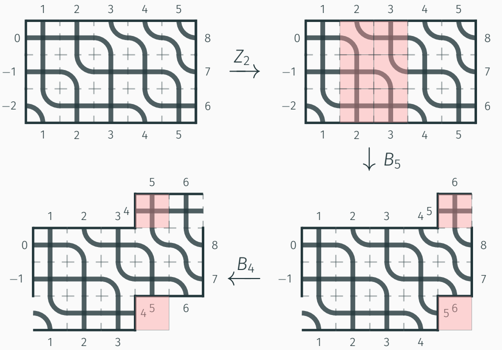

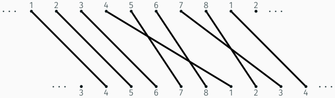

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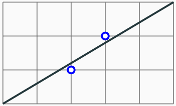

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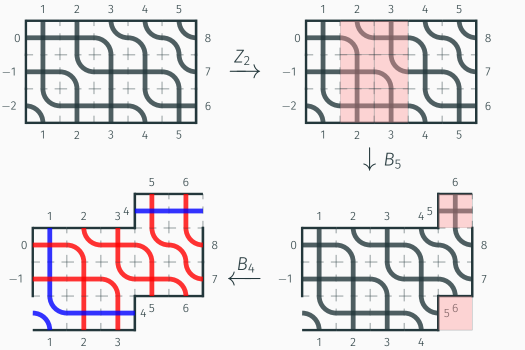

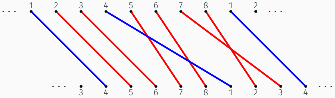

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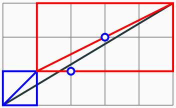

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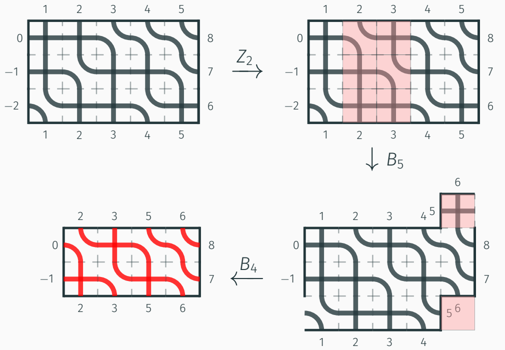

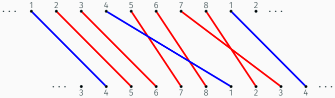

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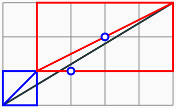

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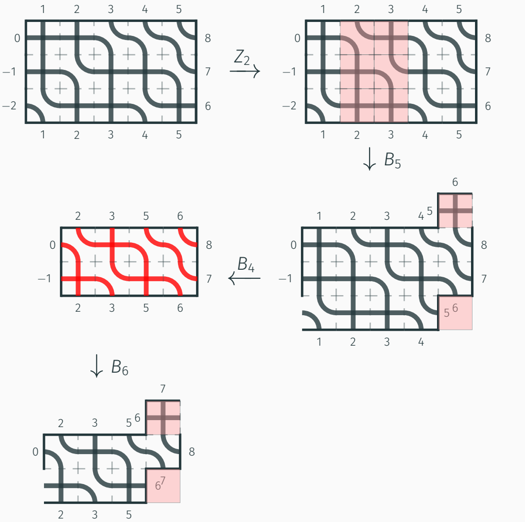

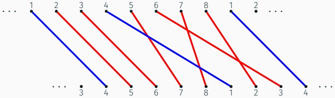

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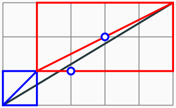

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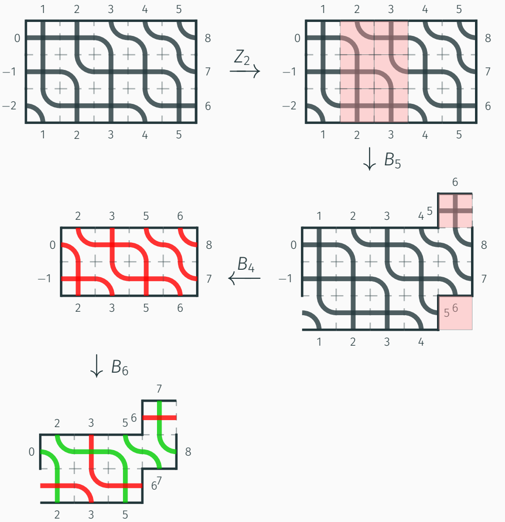

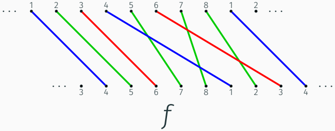

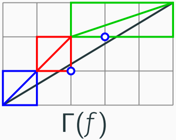

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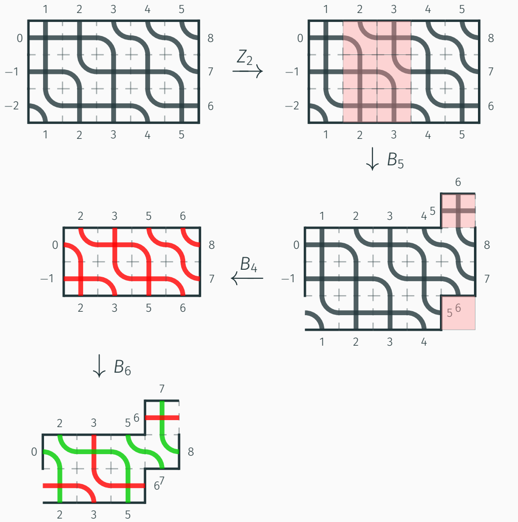
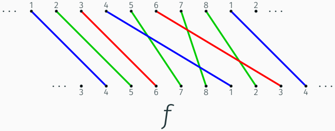
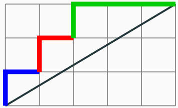






Our resulting Dyck path

Theorem Restated

To recap, we have sketched the proof of the following theorem.

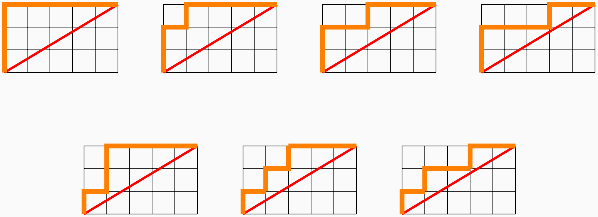
Theorem (M., 25+)
For any $0 < k < n$ with $\gcd(k, n)$ and f a repetition-free bounded affine permutation, we have a bijection

$$\mathrm{Deo}(f) \to \mathrm{Dyck}(\Gamma(f)).$$

BONUS: q,t-Rational Catalan Numbers

Rational Catalan Numbers: For $1 \leq k \leq n$ with $\gcd(k,n)=1$, the number of Dyck paths inside a $k \times (n-k)$ rectangle is counted by

$$C_{k,n-k} = \frac{1}{n}\binom{n}{k}.$$



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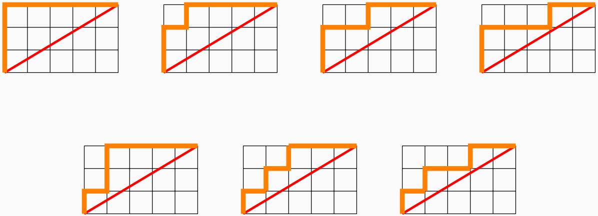

q, t**-Rational Catalan Numbers**

$$C_{k,n-k}(q,t) = \sum_{P \in \mathbf{Dyck}_{k,n-k}} t^{\text{area}} q^{\text{dinv}(P)}.$$

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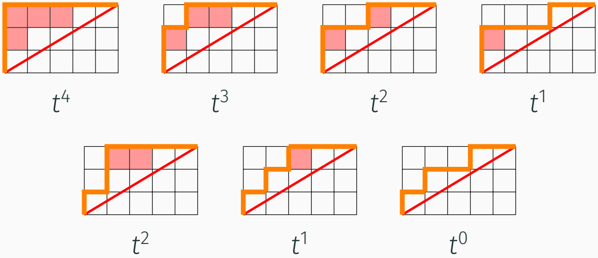

q, t**-Rational Catalan Numbers**

$$C_{k,n-k}(q,t) = \sum_{P \in \mathbf{Dyck}_{k,n-k}} t^{\mathrm{area}(P)} q^{\mathrm{dinv}(P)}.$$

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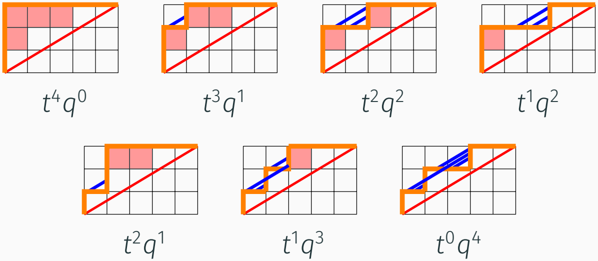

q, t-Rational Catalan Numbers

$$C_{k,n-k}(q,t) = \sum_{P \in \mathbf{Dyck}_{k,n-k}} t^{\text{area}} q^{\text{dinv}(P)}.$$

BONUS: Geometric Motivation

Grassmannian

$$\mathrm{Gr}(k,n;\mathbb{F}) := \{V \subseteq \mathbb{F}^n \mid \dim(V) = k\} = \frac{\{k \times n \text{ matrices of rank } k\}}{(\text{row operations})}.$$

BONUS: Geometric Motivation

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Theorem (Knutson-Lam-Speyer, 2013)
For bounded affine permutations f, we have a stratification

$$\mathrm{Gr}(k,n) = \bigsqcup_{f} \Pi_f^\circ,$$

into **open positroid varieties**.

BONUS: Polynomials

We have a unique "top cell" (largest dimension) open positroid variety, denoted $\Pi^{\circ}_{k,n}$.

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For $\gcd(k,n) = 1$, we may write the **mixed Hodge polynomial** of $\Pi^\circ_{k,n}$, $\mathcal{P}(\Pi^\circ_{k,n}; q, t) \in \mathbb{N}\left[q^{\frac{1}{2}}, t^{\frac{1}{2}}\right]$ as

$$\mathcal{P}(\Pi^\circ_{k,n}; q, t) = (q^{\frac{1}{2}} + t^{\frac{1}{2}})^{n-1} C_{k,n-k}(q,t).$$

BONUS: Polynomials

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Additionally, we have the point count as

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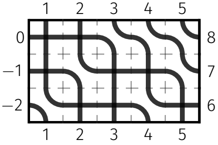
$$\begin{aligned} \#\Pi^{\circ}_{k,n}(\mathbb{F}_q) &= (q-1)^{n-1} C_{k,n-k}(q) = (q-1)^{n-1} \frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \\ &= \sum_{D \in \mathbf{Deo}_{k,n}} (q-1)^{\#\text{elbows}(D)} q^{\#\text{crossings}(D)/2}. \end{aligned}$$

These are due to (Deodhar, 1985) and (Galashin-Lam, 2021).

Thank you!



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