Coxeter Groups and Root Systems

Disclaimer: These notes are incomplete! Some proofs may be missing, but they are mostly pulled from [1], [2], [3].

1 Lecture 1 – 04/01, Thomas

(Notes by Olha)

1.1 Reflection groups ([3], 1.1)

Let V be a Euclidean space equipped with $(\cdot,\cdot)$.

Definition 1.1. *For $\alpha \in V \setminus \{0\}$, denote $\boldsymbol{H_\alpha}$ to be the hyperplane to be perpendicular to α and $\boldsymbol{L_\alpha}$ to be the line passing through α. Also denote $\boldsymbol{s_\alpha}$ to be the reflection around H_α. That is*

$$s_\alpha(\lambda) = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha.$$

Definition 1.2. *W is called a **finite reflection group** if it is a finite group generated by reflections.*

It turns out that finite reflection groups are a partial case of Coxeter groups that we will define later:

Theorem 1.3. *The finite Coxeter groups are exactly the finite reflection groups.*

1.2 Root systems ([3], 1.2)

Let W be a finite reflection group.

Proposition 1.4. *Suppose $t \in O(V)$ be an orthogonal map and $\alpha \in V \setminus \{0\}$. Then*

$$s_{t\alpha} = ts_\alpha t^{-1}.$$

In particular for $w \in W$, if $s_\alpha \in W$, then $s_{w\alpha} \in W$ as well.

Proof. We need to show that $ts^\alpha t^{-1}$ sends $t\alpha$ to its negative and fixes $H_{t\alpha}$ pointwise:

1. $ts_\alpha t^{-1}(t\alpha) = ts_\alpha(\alpha) = -t\alpha$.
2. Suppose $t\lambda \in H_{t\alpha}$. Then $\lambda \in H_\alpha$. Therefore

$$ts_\alpha t^{-1}(t\lambda) = ts_\alpha(\lambda) = t\lambda.$$

(since s_α fixes H_α).

□

Notice that for $w \in W$, we have

$$w(L_\alpha) = L_{w\alpha}$$

for every $\alpha \neq 0$. In other words W permutes the lines L_α. That is, if we take the collection of the normalized vectors α (where α ranges over the set of all reflections in W), it will be stable over the actions of W. We generalize it to the following definition

Definition 1.5. *A collection of nonzero vectors Φ is called a **root system** if:*

(1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ *for all* $\alpha \in \Phi$;

(2) $s_\alpha \Phi = \Phi$ *for all* $\alpha \in \Phi$.

*Additionally, we call Φ **crystallographic** if*

(3) $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Phi$.

If Φ *is a root system, we define the corresponding* ***associated reflection group*** W *to be generated by all reflections* $s_\alpha, \alpha \in \Phi$.

Definition 1.6. *For a root system* Φ*, define its* ***rank*** *to be*

$$\operatorname{rank}(\Phi) = \dim(\operatorname{span} \Phi).$$

2 Lecture 2 – 04/07, Ariana

(Notes by Olha)

2.1 Positive and simple systems ([3], 1.3)

Definition 2.1. *We say that the vector space* V *is* ***totally ordered*** *it is equiped with an total order relation satisfying:*

1. *For every* $\lambda, \mu, \nu \in V$*, if* $\mu < \nu$*, then* $\lambda + \mu < \lambda + \nu$.
2. *For* $\mu < \nu$ *and* $c \neq 0$*, we have* $c\mu < c\nu$ *if* $c > 0$ *and* $c\nu < c\mu$ *if* $c < 0$.

Definition 2.2. *For a root system* Φ, $\Pi \subset \Phi$ *is a* ***positive system*** *if*

$$\Pi := \{\alpha \in \Phi | \alpha > 0\}.$$

for some total ordering.

Definition 2.3. *A collection* $\Delta \subset \Phi$ *is called a* ***simple system*** *if*

1. Δ *is a basis of* $\operatorname{span}(\Phi)$*;*
2. *For every* $\alpha = \sum_{\beta \in \Delta} c_\beta \beta \in \Phi$*, all* c_β *have the same sign.*

Theorem 2.4. *There is a correspondance between simple systems and positive systems in* Φ.

1. *If* Δ *is a simple system in* Φ*, there is a unique positive system* Π *containing* Δ.
2. *If* Π *is a positive system, then it contains a unique simple system.*

Proof. The proof of one of the directions seems long, so we did not talk about it in the meeting. □

2.2 Conjugacy of positive and simple systems ([3],1.4)

Proposition 2.5. *Suppose* Δ *be a simple system, and* $\Pi \supset \Delta$ *be the corresponding positive system. Then for any* $\alpha \in \Delta$,

$$s_\alpha(\Pi \setminus \{\alpha\}) = \Pi \setminus \{\alpha\}.$$

Proof. Suppose $\beta = \sum_{\gamma \in \Delta} c_\gamma \gamma \in \Pi \setminus \{\alpha\}$. Then all $c_\gamma \geq 0$. Moreover, since $\beta \neq \alpha$, one of $c_\gamma \neq 0$.

Since s_α is a reflection with respect to H_α, $s_\alpha \beta = \beta - c \cdot \alpha$, so c_α is the only coefficient that changes. Therefore $s_\alpha \beta$ is still positive (and it is clearly not equal to α), so $s_\alpha \beta \in \Pi \setminus \{\alpha\}$. □

2.3 Length of an element ([3], 1.5-1.8)

For a simple system Δ, we will say that **simple reflections** are the ones in Δ. Denote the set of simple reflections by $\boldsymbol{S}$.

Theorem 2.6. W *is generated by the simple reflections* s_α.

Proof. Did not prove in the meeting. □

This allows us to introduce the following definition:

Definition 2.7. *The* ***length*** *of $w \in W$ is the smallest $r = \boldsymbol{\ell(w)}$ such that w can be written as $w = s_1 \dots s_r$, where $s_i \in S$.*

Define also

$$\boldsymbol{n(w)} = |\Pi \cap (w^{-1})(-\Pi)|,$$

i.e. $n(w)$ is the number of positive roots that are turned negative by w.

Theorem 2.8. *For every $w \in W$, we have*

$$\ell(w) = n(w).$$

Proposition 2.9. *For every $w \in W$*

$$\ell(w) = \ell(w^{-1})$$

and

$$n(w) = n(w^{-1}).$$

Proof. For ℓ, just notice that if $w = s_1 \dots s_r$, then $w^{-1} = s_r \dots s_1$.

For n, we need to apply the definition:

$$n(w) = |\Pi \cap w^{-1}(-\Pi)| = |-w(\Pi \cap w^{-1}(-\Pi))| = \\ |(w^{-1})^{-1}(-\Pi) \cap \Pi| = n(w^{-1}).$$

□

Lemma 2.10. *Suppose $w \in W$ and $\alpha \in \Delta$. Then:*

1. *If $w\alpha > 0$, then $n(ws_\alpha) = n(w) + 1$;*
2. *If $w\alpha < 0$, then $n(ws_\alpha) = n(w) - 1$.*

Proof. Recall that $s_\alpha \Pi = \Pi - \{\alpha\} \cup \{-\alpha\}$, so α keeps the sign of all the positive roots except for α. Those roots are then going to be turned negative either by both w and ws_α or by neither of them. We just need to look at α:

1. If $w\alpha > 0$, then $ws_\alpha(\alpha) = w(-\alpha) < 0$, so α is turned negative by ws_α, but not w;
2. If $w\alpha < 0$, then $ws_\alpha(\alpha) > 0$, so α is turned negative by w, but not by ws_α.

□

Corollary 2.11. *$n(w) \leq \ell(w)$.*

Theorem 2.12 (Deletion condition)**.** *Let $\Delta \subset \Phi$ be a simple system. Let $w = s_1, \dots, s_r$ (where $s_i = s_{\alpha_i}$, $\alpha_i \in \Delta$).*

If $n(w) < r$, then there are i, j, $1 \leq i < j \leq r$ such that

$$w = s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r.$$

Proof. 1. Show that there exist $i < j$ such that

$$\alpha_i = (s_{i+1} \dots s_{j-1})\alpha_j.$$

Since $n(w) < r$, by 2.10 there exists j such that the addition of s_j decreased $n(w)$. Therefore $(s_1 \dots s_{j-1})\alpha_j < 0$. So take $i < j$ to be the first index such that $s_i(s_{i+1} \dots s_{j-1})\alpha_j < 0$ and $(s_{i+1} \dots s_{j-1})\alpha_j >$

0. But that means that the positive root $(s_{i+1}\dots s_{j-1})\alpha_j$ is turned negative by s_i. But the only positive root turned negative by s_i is α_i, so

$$(s_{i+1}\dots s_{j-1})\alpha_j = \alpha_i.$$

2. Show that

$$s_{i+1}\dots s_j = s_i \dots s_{j-1}.$$

Denote $w' = s_{i+1}\dots s_{j-1}$. We have showed above that $w'\alpha_j = \alpha_i$. Applying $w's_j w'^{-1} = s_{w'\alpha_j}$, we get:

$$(s_{i+1}\dots s_{j-1})s_j(s_{j-1}\dots s_{i+1}) = s_{\alpha_i}.$$

This shows the desired equality.

3. Finally, rearranging the terms in the previous equality, we get:

$$s_{i+1}\dots s_{j-1} = s_i \dots s_j,$$

which concludes the proof.

□

Corollary 2.13. $n(w) = \ell(w)$.

Theorem 2.14 (Exchange condition)**.** *Suppose $w = s_1 \dots s_r$ and $\ell(ws\alpha) < \ell(w)$ for some $\alpha \in \Delta$.*

Then there exists $i \in [r]$ such that $q = s_1 \dots \widehat{s_i} \dots s_r s_\alpha$.

Proof. It is enough to recreate the proof of the Deletion condition for $s_1 \dots s_r s_\alpha$ and $\alpha_j = \alpha$. □

Theorem 2.15. *Let Δ be a simple system, corresponding to the positive system Π. For $w \in W$, the following conditions are equivalent:*

1. $w\Pi = \Pi$;
2. $w\Delta = \Delta$;
3. $n(w) = 0$;
4. $\ell(w) = 0$;
5. $w = 1$.

Notice also that the unique element changing all positive roots to all negative roots is the longest element of length $\ell(w_0) = |\Pi|$.

2.4 Coxeter system ([3], 1.9)

Theorem 2.16. *Let Δ be a simple system. For every $\alpha, \beta \in \Delta$, let $m(\alpha, \beta)$ be the order of $s_\alpha s_\beta$ in W.*

Then

$$W = \langle s_\alpha, \alpha \in \Delta | (s_\alpha s\beta)^{m(\alpha, beta)} = 1\rangle.$$

Definition 2.17. *Any group W with such a presentation (where $w(\alpha, \alpha) = 1$ as in the theorem is called a **Coxeter group**. If S is the set of generators of W, (W, S) is called a **Coxeter system**.*

3 Lecture 3 – 04/22, Matty

(Notes by Olha)

3.1 Parabolic Subgroups

Fix Φ, Π, Δ, W and S (the set of simple reflections s_α, $\alpha \in \Delta$).

Definition 3.1. *For $I \subset S$, denote*

$$\boldsymbol{W_I} := \langle s_\alpha | \alpha \in I \rangle \subset W$$

and

$$\boldsymbol{\Delta_I} := \{\alpha \in \Delta | s_\alpha \in I\}.$$

We say that $H \subset W$ is a ***parabolic subgroup*** *if there exists $I \subset S$ such that $H = W_I$.*

Notice a couple of properties of W_I. First, $W_\varnothing = \{1\}$ and $W_S = W$. Also, if Δ is replaced by another simple system $w\Delta$, then W_I would turn into wW_Iw^{-1} and Δ_I would be replaced with $w\Delta_I$.

Proposition 3.2. *For $I \subset \Delta$, define also $V_I = \operatorname{span}(\Delta_I)$ and $\Phi_I = \Phi \cap V_I$. Then:*

1. *Φ_I is a root system in V_I with symple root system Δ_I and $W_I|_{V_I}$.*
2. *If ℓ_I is the length function on W_I, then $\ell_I = \ell$ on W_I.*
3. *If we define $W^I := \{w \in W | \ell(ws) > \ell(w) \text{ for all } s \in I\}$, then for every $w \in W$, there exist unique $u \in W^I, v \in W_I$ such that $w = uv$. Moreover, $\ell(w) = \ell(u) + \ell(v)$ and u is the unique element of smallest length in wW_I.*

4 Lecture 4 – 04/29, Robert

(Notes by Robert)

4.1 Classification of finite reflection groups

Let (W, S) be a Coxeter system.

Definition 4.1. *The* ***Coxeter graph*** *Γ is defined on the vertex set S with vertices $i, j \in S$ connected by an edge if $m_{ij} \geq 3$ (i.e. s_i and s_j do not commute).*

Define also the ***weight function*** *$m : (i, j) \mapsto m_{i,j}$.*

Remark 4.2. *Since all simple systems are conjugate, the Coxeter graph does not depend on the choice of S, only on the Coxeter group W.*

Example 4.3. *With the presentation $S_{n+1} = \langle s_i = (i, i+1) \mid (s_is_{i+1})^3 = s_i^2 = e \rangle$ we get the Coxeter graph*

$$S_{n+1} \qquad \bullet \overset{3}{\text{———}} \bullet \overset{3}{\text{———}} \bullet \overset{3}{\text{———}} \cdots \overset{3}{\text{———}} \bullet \overset{3}{\text{———}} \bullet$$

With the presentation $D_m = \langle s, r \mid s^2 = r^2 = (sr)^m = e \rangle$ we get the Coxeter graph

$$D_m \qquad \bullet \overset{m}{\text{———}} \bullet$$

Definition 4.4. *A reflection group W acting on a vector space V is* ***essential*** *if it has no fixed points.*

Proposition 4.5. *Suppose W_1, W_2 are two finite reflection groups on vector spaces V_1 and V_2 which are both essential.*

If Γ_1 and Γ_2 are isomorphic, then there is a isometry $\varphi : V_1 \to V_2$ which induces an isomorphism $W_1 \to W_2$.

Proof. First, show that there is an isomorphism between V_1 and V_2. For that, recall that $S_i = \Delta_i$ is a basis of V_i. Then $\phi : \Delta_1 \to \Delta_2$ (the graph isomorphism) extends linearly to an isomorphism $\phi : V_1 \to V_2$.

Now, show that ϕ extends to an isomorphism between W_1 and W_2. For that, we will show that ϕ preserves angles.

Let $\alpha \neq \beta \in \Delta_1$. The angle between α and β is

$$\theta = \pi - \frac{\pi}{m(\alpha, \beta)}.$$

In particular, $\langle \alpha, \beta \rangle = \cos(\theta) = -\cos(\frac{\pi}{m((\alpha,\beta)})$ Therefore, since ϕ preserves m, it will preserve the angles as well. □

Definition 4.6. *For a Coxeter graph* Γ*, define the* ***associated*** $n \times n$ ***matrix*** A*(symmetric):*

$$A(s, s') = -\cos(\frac{\pi}{m(s, s')}).$$

Notice that if Γ comes from a reflection group W, then the corresponding matrix A is positive definite, since it is the Gram matrix of the root system. But it might be the case that we have some Γ which satisfies all of the conditions to be a Coxeter graph, but does not come from a reflection group, and whose associated form is not positive definite.

Definition 4.7. *A Coxeter system is* ***irreducible*** *if its graph* Γ *is connected.*

Proposition 4.8. *Suppose the graph* Γ *of a Coxeter system* (W, S) *has connected components* $\Gamma_1, \ldots, \Gamma_r$ *and associated generators* $S_1, \ldots, S_r$*.*

Then

$$W = W_{S_1} \times \ldots \times W_{S_r}$$

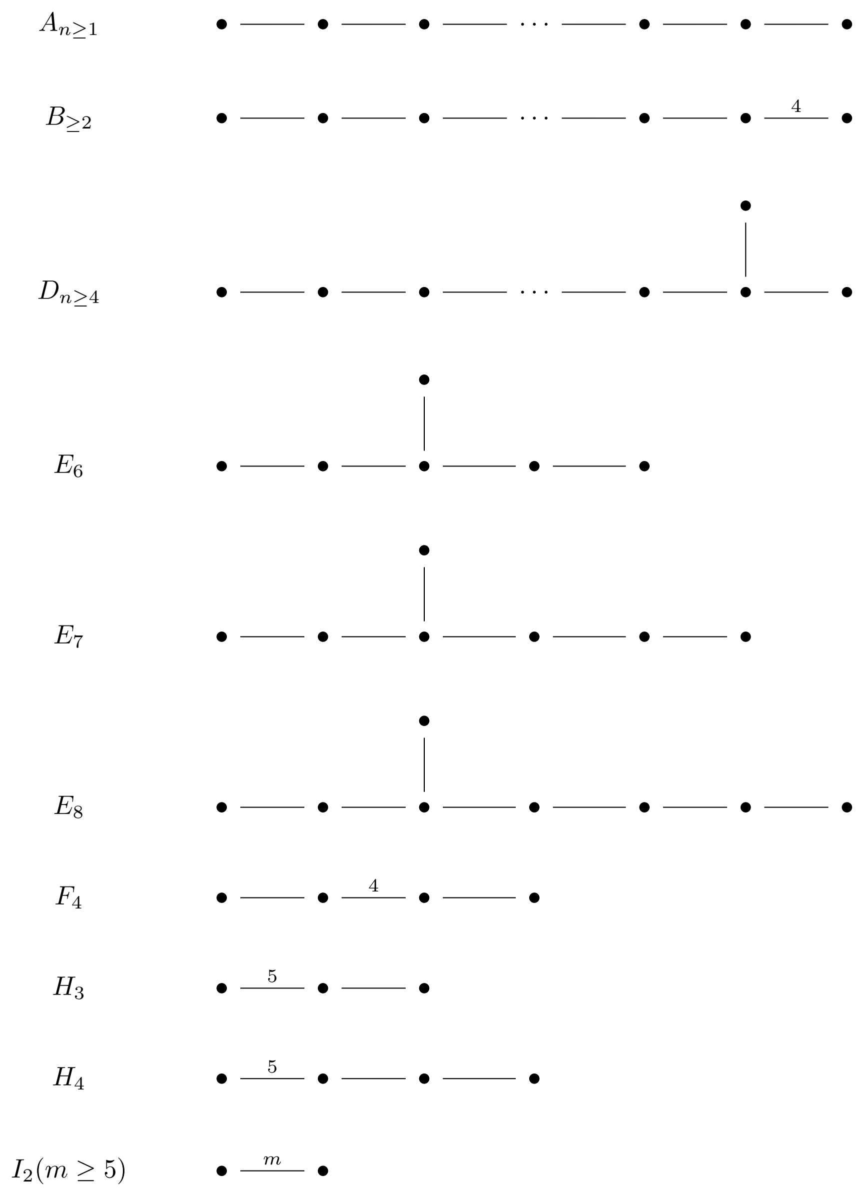
and (W_{S_i}, S_i) *is an irreducible Coxeter system.*

Proof. We proceed by induction on r, the number of connected components. Since there are no edges in the Coxeter graph, the sets S_i and S_j stabilize each other, so the parabolic subgraphs $W_{S_i} = \langle S_i \rangle$ and $W_{S_j} = \langle S_j \rangle$ stabilize each other. And the product of all W_{S_i}'s contains the set of generators S, so the product of the parabolic subgroups is all of W. So we just need to show that the product is direct. This is where we use the induction hypothesis to show that

$$W_{S \setminus S_i} = \prod_{j \neq i} W_{S_j}$$

and now we claim that W_{S_i} intersects $W_{S \setminus S_i}$. Thus the product is direct. □

Theorem 4.9. *All the Coxeter graphs represented on the picture have positive definite matrices.*



Proof. For each minor of A for Γ shown above is the determinant $\det(A')$ for another Γ', also shown above. So we just need to show that each $\det(A) > 0$. For technical reasons, we will show $\det(2A) > 0$, which is clearly equivalent.

When n is finite, we can do a direct computation. For example,

$$I_2(m) \quad \det(2A) = \det \begin{pmatrix} 2 & -2\cos(\pi/m) \\ -2\cos(\pi/m) & 2 \end{pmatrix} = 4\sin^2(\pi/m) > 0$$

Now for $n \geq 3$, we can reduce the computation as follows. Ordering the vertices $\{1, 2, \ldots, n\}$, we can choose the vertex n such that n is only connected to $n-1$, and is labeled with $m = 3$ or $m = 4$. Then the matrix $2A$ has the form

$$2A = \begin{pmatrix} 2A_{n-2\times n-2} & * & 0 \\ * & 2 & -2\cos(\pi/m) \\ 0 & -2\cos(\pi/m) & 2 \end{pmatrix}$$

Computing the cofactor expansion, if d_i is the $i \times i$ principal minor of $2A$, then we have

$$\det 2A = 2d_{n-1} - cd_{n-2}$$

Where $c = 1$ if $m = 3$ (since $\cos(\pi/3) = \frac{1}{2}$) and $c = 2$ if $m = 4$ (since $\cos(\pi/4) = \frac{1}{\sqrt{2}}$). From this, we can compute that all determinants are positive. □

Definition 4.10. $\Gamma' \subset \Gamma$ *is a* ***Coxeter subgraph*** *if either it is a "proper" (usual) subgraph or* $m'_{ij} < m_{ij}$.

Corollary 4.11. *Suppose* Γ *is a connected Coxeter graph with positive semi-definite associated bilinear form. Then every proper subgraph of* Γ' *is positive-definite.*

5 Lecture 5 – 05/06, Rushil

(Notes by Olha)

5.1 Coxeter poset

Suppose Φ is a root system with corresponding set of positive roots Π and simple system Δ. Denote also $V = \text{span}(\Delta)$.

Definition 5.1. *The* ***root poset*** *is defined by the following relations on positive roots: we say that* $\beta \leq \gamma$ *if* $\gamma - \beta$ *is a non-negative combination of simple roots.*

This order can also be viewed as the dominance order on the coordinates in the basis Δ.

Examples: A_4 and B_3 – (From Evan Chen's notes, [2])

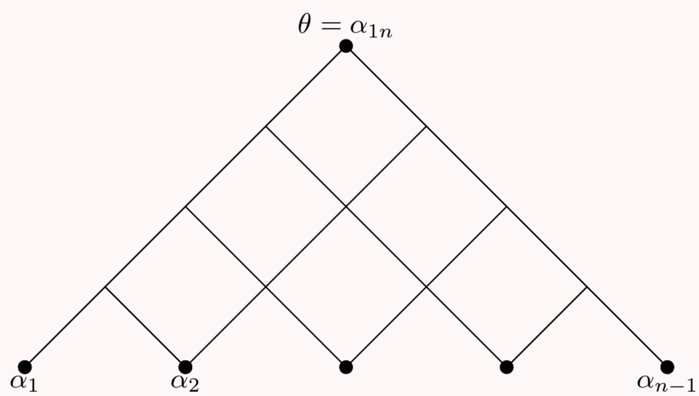
Example 13.3 (Root poset for A_{n-1})

Take $W = A_{n-1}$. Then

$$\Phi^+ = \{\alpha_{ij} = e_i - e_j\}$$

with simple roots $\alpha_i = \alpha i, i+1$. Then $\alpha_{ij} \geq \alpha_{i'j'} \iff i \leq i' < j' \leq j$.

Here is a picture of the poset for $n = 5$:

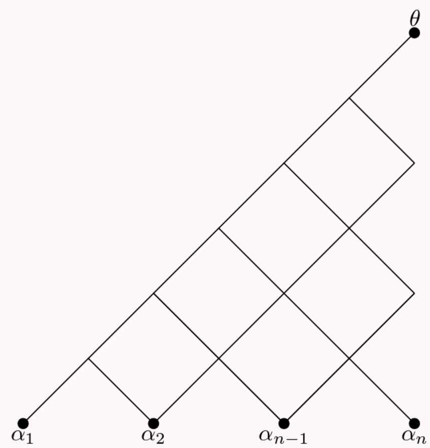


Example 13.5 (Root poset for B_n)

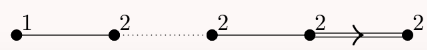
In this example we have

$$\Phi^+ = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \cup \{e_i \mid 1 \leq i \leq n\}$$

The simple roots are $\alpha_i = e_i - e_{i+1}$ and $\alpha_n = e_n$. Here is the picture of the poset for $n = 3$; it is "half" the A_n picture.



In the context of Kostant game, θ corresponds to the following endpoint.



Definition 5.2. *Define the **height** of* $\beta = \sum_{\alpha\in\Delta} c_\alpha \alpha \in \Pi$ *as*

$$\text{height}(\beta) = \sum_{\alpha\in\Delta} c_\alpha.$$

Then the poset is graded by height. We also introduce some additional notation:

Definition 5.3. *Denote* r_t *to be the number of elements of height* t *and* θ *to be the element of largest height.*

*Also introduce the **Coxeter number** of the system*

$$h = \text{height}(\theta) + 1.$$

And denote $a_1, a_2, \ldots, a_n$ *to be the coeffitients of* θ *in the basis* Δ.

We call

$$f = 1 + |\{i : a_i = 1\}|$$

*the **index of connection**.*

Proposition 5.4. *We have*

$$n = r_1 \geq r_2 \geq \ldots \geq r_{h-1} = 1.$$

Example: Let $W = A_n$. We have the positive roots $\Pi = \{e_i - e_j \mid 1 \leq i < j \leq n+1\}$ and our simple roots $\Delta = \{s_i := e_i - e_{i+1} \mid 1 \leq i \leq n\}$. The highest root is $\theta = e_1 - e_n = \sum_{i=1}^{n} s_i$. Thus, our Coxeter number is $h = \text{height}(\theta) + 1 = n + 1$, and our index of connection is $f = n + 1$.

Let $W = B_n$. Our simple roots are $\Delta = \{\alpha_i = e_i - e_{i+1} \mid 1 \leq i < n\} \cup \{\alpha_n = e_n\}$. Then, the highest root is

$$\theta = e_1 = (e_1 - e_2) + \sum_{i=2}^{n} 2\alpha_i.$$

Thus, our Coxeter number is $h = \text{height}(\theta) + 1 = 2n$, and our index of connection is $f = 2$.

Theorem 5.5. *The order of the Weyl group is:*

$$|W| = f \cdot n! \cdot a_1 \dots a_r.$$

For example, we get $|S_{n+1}| = (n+1) \cdot n! \cdot 1 = (n+1)!$.

5.2 Coxeter elements

Definition 5.6. *A* ***Coxeter element*** *is a product*

$$s_1 \cdot s_2 \cdot \dots \cdot s_n,$$

where $s_1, \dots, s_n$ are the simple reflections (in any order).

Proposition 5.7. *All Coxeter elements are conjugate.*

To show this, we will need a Lemma:

Lemma 5.8. *There is a simple reflection (without loss of generality, s_n) that commutes with all but one of the s_j.*

Proof. Notice that we can swap commuting elements and do cycle permutations since

$$s_1(s_1 \dots s_r)s_1 = s_2 \dots s_r s_1.$$

Now we proceed by induction. Suppose σ is a permutation of $[n]$. So we want to show:

$$s_1 s_2 \dots s_n$$

is conjugate to

$$s_{\sigma(1)} \dots s_{\sigma(n)}.$$

Let k be such that $\sigma(k) = n$. By induction hypothesis, we can set all the elements but s_n in the right order (since s_n commutes with everything but one element). Then, we move s_n in one of the two directions to get it to the right position. □

Proposition 5.9. *The Coxeter number h can be also represented as the order of a Coxeter element.*

Notice that every Coxeter element is an orthogonal map, it has n eigenvalues of absolute value 1.

Proposition 5.10. 1 *is not an eigenvalue of the Coxeter element.*

Definition 5.11. *Denote $e^{2\pi i \cdot \frac{\varepsilon_j}{n}}$ for $j \in [n]$ to be the eigenvalues of Coxeter element.*

Call $0 < \varepsilon_1 \leq \dots \leq \varepsilon_n < h$ the ***exponents****.*

Theorem 5.12. *$\overrightarrow{r} = (r_1 \geq \dots \geq r_{h-1})$ and $\overrightarrow{\varepsilon} = (\varepsilon_n \geq \dots \geq \varepsilon_1)$ are dual partitions.*

Proposition 5.13 (Exponents facts)**.**

- *$\varepsilon_1 = 1$ and $\varepsilon_n = h - 1$.*
- *$|W| = \prod_{j=1}^{n}(1 + \varepsilon_i)$.*
- *$|\Pi| = \frac{n \cdot h}{2}$.*

6 Lecture 6 – 05/13, Ian

(Notes by Ian)

6.1 Affine Weyl Groups

Suppose Φ is a root system, now crystallographic. Let Π, Δ be the sets of positive roots and single roots respectively. Let W be the Weyl group and $V = \text{span}(\Phi)$.

Definition 6.1. *The* ***coroot*** *for* $\alpha \in \Phi$ *is*

$$\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle}\alpha.$$

Define also

$$\mathbf{\Phi}^\vee = \{\alpha^\vee | \alpha \in \Phi\}.$$

This allows us to rewrite

$$s_\alpha \beta = \beta - \langle \beta, \alpha^\vee \rangle \alpha.$$

It also turns out that $\Phi^\vee$ is a root system as well.

Definition 6.2. *The* ***root lattice*** *is defined as*

$$Q = \text{span}_{\mathbb{Z}} \Phi \quad \textit{or, equivalently,} \quad Q = \text{span}_{\mathbb{Z}} \Delta.$$

The ***coroot lattice*** *is*

$$Q^\vee = \text{span}_{\mathbb{Z}} \Phi^\vee.$$

Definition 6.3. *The* ***fundamental weights*** *are* $\omega_1, \ldots, \omega_n \in V$ *such that*

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij},$$

where $\Delta = \{\alpha_1, \ldots, \alpha_n\}$.

Define also the ***weight lattice***

$$\mathbf{P} = \text{span}_{\mathbb{Z}}\{\omega_1, \ldots, \omega_n\}.$$

Finally, the ***coweight lattice*** $\mathbf{P}^\vee$ *is the weight lattice of the coroot system.*

Proposition 6.4. *We have*

$$Q \subset P.$$

Moreover,

$$\alpha_i = \sum_j \langle \alpha_i, \alpha_j^\vee \rangle \omega_j = \sum_j c_{ij}\omega_j,$$

where $C = (c_{ij})$ *is the Schläfli matrix, i.e.* $c_{ij} = -\cos(\pi/m_{ij})$ *where* m_{ij} *is the order of* $s_i s_j$. *It follows that*

$$[P : Q] = \det(C) = f,$$

where f *is the index of connection of the corresponding root poset.*

Now, we can define Affine Weyl groups:

Definition 6.5. *Given* $\alpha \in \Phi$ *and* $k \in \mathbb{Z}$*, define an* ***affine hyperplane***

$$H_{\alpha,k} = \{\lambda \in V | \langle \lambda, \alpha \rangle = k\}.$$

Define also the ***affine reflections***

$$s_{\alpha,k}(\lambda) = \lambda - (\langle \lambda, \alpha \rangle - k)\alpha^\vee.$$

Then the ***affine Weyl group*** *is the group* $\mathbf{W}_{\mathit{aff}}$ *generated by* $s_{\alpha,k}$ *for all* $\alpha \in \Phi, k \in \mathbb{Z}$.

Notice that $H_{\alpha,0} = H_\alpha$, and more generally that $H_{\alpha,k} = H_\alpha + \frac{k}{2}\alpha^\vee$.

Let $\mathrm{Aff}(V)$ denote the group of all orthogonal affine transformations from V to V. Notice that W and W_{aff} are subgroups of $\mathrm{Aff}(V)$. Also, we can identify each $\lambda \in Q^\vee$ with the map $t_\lambda : V \to V$ given by $t_\lambda(\mu) = \lambda + \mu$. Under this identification, $Q^\vee$ is also a subgroup of $\mathrm{Aff}(V)$.

Theorem 6.6. *$W_{aff} = W \ltimes Q^\vee$ (as subgroups of $Aff(V)$).*

Proof. Notice that W normalizes $Q^\vee$, since

$$s_\alpha t_\lambda s_\alpha^{-1} = t_{s_\alpha \lambda}.$$

And, W and $Q^\vee$ have trivial intersection, since every element of W fixes $0 \in V$, while the only element of $Q^\vee$ which fixes 0 is the identity. So, $W \ltimes Q^\vee$ is well-defined.

Note that for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$,

$$t_{\alpha^\vee} = s_{\alpha,1} s_\alpha.$$

Since the set $\{t_{\alpha^\vee} : \alpha \in \Phi\}$ generates $Q^\vee$, we have that $Q^\vee \subseteq W_{\mathrm{aff}}$. Clearly $W \subseteq W_{\mathrm{aff}}$, meaning $W'W \ltimes Q^\vee \subseteq W_{\mathrm{aff}}$. Conversely, we have

$$s_{\alpha,k} = t_{k\alpha^\vee} s_\alpha.$$

Hence every generator W_{aff} may be written as a product of an element of $Q^\vee$ and an element of W, meaning $W_{\mathrm{aff}} \subseteq W \ltimes Q^\vee$. □

Definition 6.7. *The **alcoves** are the connected components in $V - \bigcup_{\alpha \in \Phi, k \in \mathbb{Z}} H_{\alpha,k}$.*

*The **fundamental alcove** is*

$$\mathbf{A_0} = \{\lambda \in V | 0 < (\lambda, \alpha) < 1 \textit{ for all } \alpha \in \Pi\}.$$

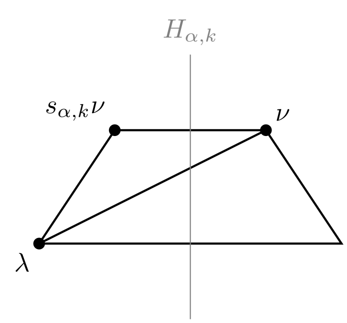
Theorem 6.8. *W_{aff} acts transitively on the alcoves.*

Proof. Let A be any alcove and A_0 the fundamental alcove. Let $\lambda \in A$ and $\mu \in A_0$.

Note that the $Q^\vee$-orbit of λ is a shift of a lattice, so is a discrete subset of V. Since $W_{\mathrm{aff}} = W \ltimes Q^\vee$ and W is a finite group, the W_{aff}-orbit of λ is a union of finitely many shifts of the $Q^\vee$-orbit, so is discrete as well. Hence, we may take $\nu = w\lambda$ whose distance to μ is minimal.

We claim that ν and μ lie in the same alcove. Since distinct alcoves are disjoint, this will imply $wA = A_0$, completing the proof.

Suppose for contradiction that ν and μ are in different alcoves, meaning they are separated by an affine hyperplane $H_{\alpha,k}$. Then, the geometric below construction shows $s_{\alpha,k}\nu$ is closer to μ than λ, since the diagonal of a trapezoid is shorter than each of the congruent sides. This contradicts the fact that the distance from ν to μ is minimal.



□

Proposition 6.9. *If $\tilde{\alpha}$ is the highest root (i.e. $\tilde{\alpha} - \alpha \in \operatorname{span}_{\mathbb{Z}_{\geq 0}} \Delta$ for all $\alpha \in \Pi$), then*

$$A_0 = \{\lambda \in V | 0 < \langle \lambda, \alpha \rangle \text{ for all } \alpha \in \Delta\} \cap \{\langle \lambda, \tilde{\alpha} \rangle < 1\}.$$

Inspired by this, the refer to the hyperplanes $\{H_{\alpha,0} : \alpha \in \Delta\} \cup \{H_{\tilde{\alpha},1}\}$ as the **walls** of A_0. The walls of any other alcove $A = wA_0$ are defined to be the images of the walls of A_0 under w. Let $S_0 = \{s_{\alpha,0} : \alpha \in \Delta\} \cup \{s_{\tilde{\alpha},1}\}$ be the reflections corresponding to the walls of A_0.

Proposition 6.10. *S_0 generates W_{aff}.*

Proof. Let W' be the subgroup of W_{aff} generated by S_0. An identical proof to the one above shows that W' acts transitively on the alcoves.

Now, let $s_{\alpha,k}$ be any generator of W_{aff}, and let A be an alcove having $H_{\alpha,k}$ as a wall. Then $A = wA_0$ for some $w \in W'$. Then $wH_{\alpha,k}$ is a wall H of A_0, which implies $ws_{\alpha,k}w^{-1} = s \in S_0$, where s is the reflection corresponding to H. Thus $s_{\alpha,k} = w^{-1}sw \in W'$. □

Definition 6.11. *The **length** of an element $w \in W_{aff}$, $l(w)$, is the smallest r for which w is a product of r elements of S_0.*

Proposition 6.12. *For $w \in W_{aff}$, let*

$$\mathscr{L}(w) = \{H_{\alpha,k} |\ H_{\alpha,k} \text{ separates } A_0 \text{ and } wA_0\}.$$

Then $l(w) = \#\mathscr{L}(w)$.

The idea of the proof is to show (by induction on the length of w), that if $w = s_1 \dots s_l$ is a minimal decomposition of w into elements of S_0 where H_i is the hyperplane corresponding to s_i, then

$$\mathscr{L}(w) = \{H_1, s_1H_2, s_1s_2H_3, \dots, s_1 \cdots s_{l-1}H_l\}.$$

Moreover, these hyperplanes are distinct from one another. The following is an easy consequence.

Theorem 6.13. *W_{aff} acts simply transitively on the alcoves.*

Proof. Suppose $w \in W_{\text{aff}}$ fixes A_0. Then $\mathscr{L}(w) = \emptyset$, meaning $l(w) = 0$, so $w = 1$. □

7 Lecture 7 – 05/20, Olha

(Notes by Thomas)

7.1 Proof of Weyl's formula

Recall that Weyl's formula 5.5 states:

$$|W| = f \cdot n! \cdot a_1 \dots a_n,$$

where a_i is the number of elements of height i and f is the number of i such that $a_i = 1$.

Here, we are going to prove this formula. For that, define:

Definition 7.1. *Define two parallelepipeds:*

$$\Pi = \{x \in V \,|\, 0 \leq (x, \alpha_i) \leq 1 \quad \forall i\}$$

and

$$H = \{x \in V \,|\, -1 \leq (x, \alpha) \leq 1 \quad \forall \alpha \in \Phi\}$$

Notice that then Π is generated by $\omega_1^\vee, \dots, \omega_n^\vee$ (since $(\omega_i^\vee, \alpha_j) = 1$). At the same time, A_0 is generated by $\omega_1^\vee/a_1, \dots, \omega_n^\vee/a_n$. Therefore, we get

$$\frac{\operatorname{vol}\Pi}{\operatorname{vol}A_0} = n!a_1 \dots a_n.$$

Now, notice that H consists of $|W|$ alcoves, so

$$\frac{\operatorname{vol}H}{\operatorname{vol}A_0} = |W|.$$

For the proof of Weyl's formula we only need to prove that

$$\frac{\operatorname{vol}H}{\operatorname{vol}P} = f.$$

Lemma 7.2. *We have that*

- Π *is the fundamental domain of the coweight lattice* $P^\vee$*.*
- H *is the fundamental domain of the coroot lattice* $Q^\vee$*.*

But then we get

$$\frac{|W|}{n!a_1 \dots a_n} = \frac{\operatorname{vol}H}{\operatorname{vol}P} = [P^\vee : Q^\vee] = f,$$

which concludes the proof of Weyl's formula.

8 Lecture 8 – 05/31, Luna

(Notes by Olha)

8.1 Group algebras

In representation theory, with a group G, a representation of G can be identified with $k[G]$-modules.

$k[G]$ is an algebra over a field k, with basis indexed by elements $g \in G$; specifically $\{b_g \mid g \in G\}$. Multiplication is defined as $b_g b_h = b_{gh}$.

For example, when $G = S_n$, our generating set $S = \{s_1, \dots, s_{n-1}\}$, we have the relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $s_i s_j = s_j s_i$ when $|j - i| > 1$.

So, $k[S_n] = \{\sum_{w \in S_n} c_w T_w \mid c_w \in k\}$. It is generated as an algebra by $\{T_i \mid 1 \leq i \leq n-1\}$, with T_i identified by our s_i's. When $w = s_1 \dots s_r$, $T_w = T_1 \dots T_r$.

We can imagine Hecke algebras as a q-analogue of group algebras.

8.2 Hecke Algebras

Definition 8.1. $\mathscr{H}_{S_n}$ *is an algebra over* $\mathbb{Z}[q, q^{-1}]$ *with basis (as a vector space) given by* T_w *for* $w \in S_n$*, and generated as an algebra by* T_i *for* $1 \leq i \leq n-1$ *where* $T_i := T_{s_i}$*, with the following relations:*

1. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,
2. $T_i T_j = T_{s_i s_j}$ *for* $i \neq j$,
3. $T_i^2 = (q-1)T_i + qT_e$ *(where* T_e *is our multiplicative identity).*

Note that as $q \mapsto 1$, we recover $k[S_n]$

Recalling definitions, we have (W, S) as our Coxeter system, where $W = \langle s_\alpha \,|\, \alpha \in S\rangle$. We denote T by our set of all reflections in W.

Note that when α is a root, we have $\alpha = w(\alpha_s) \mapsto wsw^{-1} : \alpha \mapsto -\alpha$, where α_s is a simple root for some $s \in S$.

Definition 8.2. *The Hecke algebra $\mathscr{H}$ for (W, S) is an associative $\mathbb{Z}[q, q^{-1}]$-algebra with identity elements T_1 and basis as a vector space $\{T_w \,|\, w \in W\}$, and relations*

1. $T_sT_w = T_{sw}$ *if* $\ell(sw) > \ell(w)$,
2. $T_sT_w = (q-1)T_w + qT_{sw}$ *if* $\ell(sw) < \ell(w)$.

2'. $T_s^2 = (q-1)T_s + qT_1$.

(can use 2 or 2').

$\mathscr{H}$ is generated as an algebra by T_s for $s \in S$. Take any $w \in W$, $w = s_1 \dots s_r$ reduced. Then using relation 1 iteratively,

$$T_w = T_{s_1} \dots T_{s_r}.$$

Proposition 8.3. *We also have the "right handed rules"*

a. $T_wT_s = T_{ws}$ *if* $\ell(ws) > \ell(w)$,

b. $T_wT_w = (q-1)T_w + qT_{ws}$ *if* $\ell(ws) < \ell(w)$.

Proof. Can prove this inductively on ℓ. □

Do these T_w have inverses? Using relation $2'$, we get

$$T_s[q^{-1}(T_s - (q-1)T_1)] = T_1.$$

So, $(T_s)^{-1} = q^{-1}(T_s - (q-1)T_1)$. So, when $w = s_1 \dots s_r$ is reduced,

$$(T_w)^{-1} = T_{s_r}^{-1} \dots T_{s_1}^{-1}.$$

However these are computationally cumbersome. To solve this, we introduce R-polynomials.

8.3 R-polynomials.

We first briefly recall the Strong Bruhat order.

When $w't = w$ and $\ell(w) > \ell(w')$, we write $w' \xrightarrow{t} w$ and say $w > w'$ in the Bruhat order.

Proposition 8.4.

1. *If $w > w'$ and $w = w's_\alpha$. Then letting $\beta = w'(\alpha)$, $(w')^{-1}s_\beta w' = s_\alpha$, implying $w = s_\beta w'$.*
2. *If $s_1 \dots s_r$ is reduced, we define a subexpression as $s_1 \dots \widehat{s_{i_1}} \dots \widehat{s_{i_2}} \dots \widehat{s_{i_p}} \dots s_r$. (we can remove any number of the s_i's, need not be reduced). We say $w' \leq w$ iff w' occurs as a subexpression of w.*

Proposition 8.5. $(T_{w^{-1}})^{-1} = \epsilon_w q_w^{-1} \sum_{x \leq w} \epsilon_x R_{x,w}(q) T_x$ *where $\epsilon_w = (-1)^{\ell(w)}$ and $q_w = q^{\ell(w)}$. We have* $\deg(R_{x,w}(q) = \ell(w) - \ell(x)$ *and $R_{w,w} = 1$. (Also have $R_{x,w} = 0$ for $x > w$).*

To prove this, need a technical lemma.

Lemma 8.6. *For $s \in S$ and $w \in W$ with $sw < w$. If $x < w$ then*

1. *if $sx < x$, then $sx < sw$,*
2. *if $sx > x$, then $sx \leq w$ and $x \leq sw$.*

(in either case, $sx \le w$).

Matrix of R-polynomials M, indexing the rows/columns by $w \in W$ wrt some linear extension of Bruhat order.

We know $M_{ww} = 1$ and $M_{xw} \neq 0$ iff $x \le w$. This tells us the matrix is a unipotent upper triangular matrix, with finitely many entries in each column (since every $w \in W$ has a finite reduced expression, so only finitely many subexpressions).

This implies M is invertible and has a "nice" inverse formula. However this formula is nicer once we define an involution

Definition 8.7. *Define $\iota : \mathscr{H} \to \mathscr{H}$ by $\iota(T_w) = (T_{w^{-1}})^{-1}$ and $\iota(q) = q^{-1}$*

Question: Is there a basis fixed by ι? Yes, Kazhdan-Lusztig polynomials.

9 Lecture 9 – 06/07, Andrew

(Notes by Thomas)

9.1 Combinatorics in other types

We focus slightly on type B. How can we actually "generalize" combinatorics to other types? There are some steps to follow:

1. Take a combinatorial object.
2. Define in terms of root systems (probably start with type A).
3. Ask what this means for other types.
4. **Pray**

We demonstrate by examples. Example 1:

1. Our first object is S_n.
2. This object is the reflection group of A_{n+1}.
3. For other types, this means the Weyl group of that type.
4. In type B, we get signed permutations, which are permutations of $\{-n, \dots, -1, 1, \dots, n\}$ such that $\sigma(-i) = -\sigma(i)$.

Example 2:

1. The permutohedron Π_n is the convex hull of all permutations of $(1, \dots, n)$ embedded in $\mathbb{R}^n$ (note that $\dim(\Pi_n) = n - 1$.) Let us now shift by $-\frac{1}{n}\binom{n+1}{2}(1, \dots, 1)$, so that the sum of coordinates is 0.
2.

   Proposition 9.1. *We have*
   $$\Pi_n = \frac{1}{2}\sum_{\alpha\in\Pi}[-\alpha, \alpha],$$
   where Π is the set of positive roots in A_n, and this sum is the Minkowski sum.
3. For other types, the permutohedron of that type is defined to be
   $$\Pi_n^W = \frac{1}{2}\sum_{\alpha\in\Pi_W}[-\alpha, \alpha].$$

We can also generalize Π_n via taking the convex hull of some vector under the orbit of that Weyl group. The question becomes, what is a canonical choice of that vector?

Answer: Weyl vector. Recall that the fundamental weights are $\omega_1, \ldots, \omega_r \in V$ such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$

In type A, $(e_i - e_j)^\vee = e_i - e_j$, so $\omega_k = \sum_{i=1}^k e_i - \frac{k}{n} \sum_{i=1}^n e_i$.

Definition 9.2. *The Weyl vector is* $\rho = \sum \omega_i$.

Proposition 9.3. $\rho = \frac{1}{2} \sum_{\alpha \in \Pi} \alpha$.

Example 3: Non-crossing partitions.

We can define non-crossing partitions and create a poset. It turns out this is a ranked lattice, where the rank is the number of sets, and that the number of non-crossing partitions is Catalan.

Proposition 9.4. *The number of non-crossing partitions of rank k is $N_{n,k}$ (the Narayana numbers), where*

$$\begin{aligned} N_{n,k} &= \#\ \textit{Dyck paths with } k \textit{ peaks} \\ &= h\textit{-vector of the associahedron.} \end{aligned}$$

A reflection in a Coxeter group is anything conjugate to a simple reflection. We say $L(w) =$ the smallest length of w as a product of reflections. We say $u \lessdot w$ if $L(u) + 1 = L(w)$ and $w = ut$ for some reflection t.

The non-crossing partition lattice is $[1, c]$ where c is a Coxeter word.

10 Lecture 10 – 06/13, Thomas

(Notes by Olha)

10.1 Strong Bruhat Order

Proofs here are from Björner and Brenti, [1].

Definition 10.1 (Strong Bruhat Order)**.** *For $u \le w$, say that:*

1. $u \xrightarrow{t} w$ *for* $t = u^{-1}w \in T$ *and* $\ell(u) < \ell(w)$;
2. $u \to w$ *if* $u \xrightarrow{t} w$ *for some* $t \in T$.
3. $u \le w$ *in* ***Strong Bruhat Order*** *if there is a sequence* $u \to u_1 \to u_2 \to \ldots \to w$.

Proposition 10.2. 1. *If $u \le w$, then $\ell(u) \le \ell(w)$;*

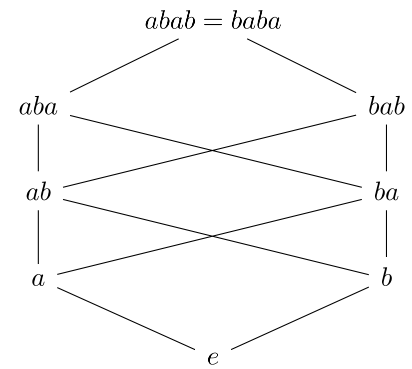
2. $u \le ut$ *iff* $\ell(u) \le \ell(ut)$;
3. *If $w = s_1 \ldots s_q$ is reduced, we have a chain*

$$e \to s_1 \to s_1 s_2 \to \ldots \to w.$$

Below is an example of the Strong Bruhat Order on $W = I_2(4) \cong B_2$, which has Coxeter graph

$$a \overset{4}{\text{———}} b$$

We have $T = \{a, b, aba, bab\}$, and we get the following Hasse diagram:



Lemma 10.3. *Suppose $u \neq w \in W$, and $w = s_1 \dots s_q$ is reduced, and suppose some reduced expression of u is a subword of $s_1 \dots s_q$. Then there exists $v \in W$ such that:*

1. *$v > u$;*
2. *$\ell(v) = \ell(u) + 1$;*
3. *Some reduced expression of v is a subword of $s_1 \dots s_q$.*

Recall:

Proposition 10.4 (Strong Exchange Condition)**.** *Suppose $w = s_1 \dots s_k$ is reduced and $t \in T$. Then the following conditions are equivalent:*

1. *$\ell(wt) < \ell(w)$;*
2. *$wt = s_1 \dots \widehat{s}_i \dots s_k$ for some i;*
3. *$t = s_k s_{k-1} \dots s_i \dots s_{k-1} s_k$.*

Proof of the Lemma. Of all reduced subexpressions

$$u = s_1 \dots \widehat{s}_{i_1} \dots \widehat{s}_{i_k} \dots s_q$$

choose one so that i_k is the smallest possible.

Let $t = s_q s_{q-1} \dots s_{i_k} \dots s_{q-1} s_q \in T$ (by definition of T). Then define

$$v = ut = s_1 \dots \widehat{s}_{i_1} \dots \widehat{s}_{i_{k-1}} \dots s_{i_k} s_q$$

(so we add s_{i_k} back to u). If $ut > u$, then we are done. So suppose $ut < u$. Then by part 3 of the Strong Exchange Condition, we have:

$$t = s_q s_{q-1} \dots s_p \dots s_{q-1} s_q \text{ for } p > i_k$$

or

$$t = s_q \dots \widehat{s}_{i_k} \dots \widehat{s}_{i_d} \dots s_r \dots \widehat{s}_{i_d} \dots \widehat{s}_{i_k} \dots s_q \text{ for } r > i_k, r \neq i_j.$$

For the first case, we get

$$\begin{aligned} w &= wt^2 \\ &= (s_1 \dots s_{i_k} \dots s_q)(s_q \dots s_{i_k} \dots s_q)(s_q \dots s_p \dots s_q) \\ &= s_1 \dots \widehat{s}_{i_k} \dots \widehat{s}_p \dots s_q \end{aligned}$$

which implies $q = \ell(w) < q$. In the second case, we get

$$\begin{aligned} u &= ut^2 \\ &= (s_1 \dots \widehat{s}_{i_1} \dots \widehat{s}_{i_k} \dots s_q)(s_q \dots s_{i_k} \dots s_q)(s_q \dots \widehat{s}_{i_k} \dots \widehat{s}_{i_d} \dots s_r \dots \widehat{s}_{i_d} \dots \widehat{s}_{i_k} \dots s_q) \\ &= s_1 \dots \widehat{s}_{i_1} \dots \widehat{s}_r \dots s_{i_k} \dots s_q, \end{aligned}$$

which contradicts the minimality of i_k. □

Proposition 10.5 (The Subword property)**.** *For $u, w \in W$, the following are equivalent:*

1. *$u \leq w$ in Strong Bruhat Order;*
2. *For every reduced expression of w, there is a subword that is a reduced word for u;*
3. *There exists a reduced expression of w, that has a subword that is a reduced word for u.*

Proposition 10.6 (The Chain property)**.** *If $u < w$, there exists a chain*

$$u = x_0 < x_1 < \dots < x_t = w$$

such that $\ell(x_i) = \ell(u) + i$.

Proposition 10.7 (Lifting property)**.** *Suppose $u < w$, $su > u$ and $sw < w$.*

Then $u \leq sw$ and $su \leq w$.

Proof. We let $\alpha \prec \beta$ denote the subword relation between a word β and a subword α. Let $sw = s_1 s_2 \dots s_q$ be a reduced decomposition. Then, $w = ss_1 s_2 \dots s_q$ is also reduced, and by the Subword Property, there exists a subword $u = s_{i_1} \dots s_{i_k} \prec ss_1 s_2 \dots s_q$. Since $u < su$, $s_{i_1} \neq s$, thus

$$s_{i_1} \dots s_{i_k} \prec s_1 s_2 \dots s_q \implies u \leq sw$$

and

$$ss_{i_1} \dots s_{i_k} \prec ss_1 s_2 \dots s_q \implies su \leq w.$$

□

10.2 Weak Bruhat Order

Definition 10.8. *For $u, w \in W$, say that:*

- *$u \leq_R w$ if*

  $$w = us_1 \dots s_k$$

  for some $s_i \in S$ such that $\ell(us_1 \dots s_i) = \ell(u) + i$.
- *$u \leq_L w$ if*

  $$w = s_k \dots s_1 u$$

  such that $\ell(s_i \dots s_1 u) = \ell(u) + i$.

These two orders are typically not (literally) equivalent, as they were for the strong order, but the two orders are isomorphic under the map $w \mapsto w^{-1}$.

Proposition 10.9 (Properties)**.**

1. *There exists a one-to-one correspondence between the reduced decompositions of w and the maximal chains in the interval $[e, w]$.*
2. *$u \leq_R w$ iff $\ell(u) + \ell(u^{-1}w) = \ell(w)$.*
3. *The weak order satisfies the prefix property. That is, $u \leq_R w$ iff there is a reduced expression $u = s_1 \dots s_k$ such that $w = s_1 \dots s_k s'_1 \dots s'_q = us'_1 \dots s'_q$.*
4. *The weak order satisfies the analogous chain property.*

5. *The weak order is a complete meet-semilattice.*
6. *If W is finite, we get a lattice.*

. . .

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