Coxeter Groups and Root Systems Syllabus

Today 2024

This seminar will provide an introduction to the theory of Coxeter groups from an algebraic combinatorial point of view. We plan to work towards the finite classification of finite root systems and finite crystallographic Coxeter groups, and understand the structure of irreducible Coxeter groups. These objects and results have many connections to other areas, like the classification of cluster algebras of finite type and representations of quivers. We will meet twice a week, on Mondays and Fridays.

The general flow will be for a talk on Monday and an exercise session on Friday. The Friday session may include some additional material/definitions, but is completely optional. Everything discussed on Friday will not be assumed for the Monday talks. The speaker is in charge of the Monday talk and selecting a few problems for the following exercise session (though they need not be present for the exercise session).

We will roughly follow [1], [2], [3], [4], [5], [6], as deemed fit.

1. **Introduction to Root Systems and Reflection Groups**
   Summary: We begin with motivating examples and understand why root systems and Coxeter groups are studied. We state some key results which we will prove later in the course. We then introduce reflection groups, (crystallographic) root systems, and Weyl groups. Time permitting, we define simple, positive, and negative roots.
   Date: Monday, April 1, 2024, **Speaker:** Thomas.

   Exercise session: Suggested exercises: Date: Friday, April 5, 2024
   References: [4] Sections 1.1 - 1.3

2. **The Basics of Finite Reflection Groups I**
   Summary: We continue where left off. We prove some properties of simple systems. We define the length function and the deletion and exchange conditions of reflection groups. We then discuss the simple transitivity of the permutation action of the Weyl group on the positive roots.
   Date: Monday, April 8, 2024, **Speaker:** Ariana.

   Exercise session:. Suggested exercises:.Date: Friday, April 12, 2024
   References: [4] Sections 1.4 - 1.8

3. **No Meeting – IPAM Workshop II**

4. **The Basics of Finite Reflection Groups II**
   Summary: We continue with our exploration of finite reflection groups by defining Coxeter systems. We define parabolic subgroups, Poincaré polynomials, fundamental domains. We show parabolic subgroups form a lattice.
   Date: Monday, April 22, 2024, **Speaker:** Matty, tentatively.

   Exercise session:. Suggested exercises:.Date: Friday, April 26, 2024
   References: [4] Sections 1.9 - 1.14

5. **Classification of Finite Reflection Groups**
   Summary: We discuss the classification of finite reflection groups and define a bilinear form associated to Coxeter graphs. We explore these finite reflection groups and understand important properties.
   Date: Monday, April 29, 2024, **Speaker:** Robert, tentatively.

Exercise session:. Suggested exercises:.Date: Friday, May 3, 2024
References: [1] Sections 2.3, 2.4.

6. **Affine Reflection Groups I**
   Summary: We define affine reflection groups, affine Weyl groups, alcoves, walls, and the length function. We show that the affine Weyl group acts simply transitively on alcoves.
   Date: Monday, May 6, 2024, **Speaker:**

   Exercise session:. Suggested exercises:. Date: Friday, May 10, 2024
   References: [4] Sections 4.1 - 4.5

7. **Affine Reflection Groups II**
   Summary: We understand affine reflection groups via their Coxeter graphs and derive a similar classification result. We find a formula for the order of Weyl groups.
   Date: Monday, May 13, 2024, **Speaker:** Ian

   Exercise session:. Suggested exercises:. Date: Friday, May 17, 2024
   References: [4] 4.6 - 4.10,

8. **Coxeter Groups and the Strong Bruhat order**
   Summary: We sketch how one may show all finite Coxeter systems are finite reflection groups. We define the strong Bruhat order
   Date: Monday, May 20, 2024, **Speaker:**

   Exercise session: We go through some examples more closely. We prove rank 2 roots systems are dihedral.
   Suggested exercises:. Date: Friday, May 24, 2024
   References: [1] 2.1 - 2.3, [4] 6.4

9. **The Weak Bruhat Order**
   Summary: We define the weak Bruhat order and show it is a meet semilattice. We also show some statements regarding nil- and braid-moves for reducing expressions of words in a Coxeter group.
   Date: Monday, May 27, 2024, **Speaker:**

   Exercise session:. Suggested exercises:. Date: Friday, May 31, 2024
   References: [1] 3.1 - 3.3

10. **TBD**
    Summary: We can go over any extra topic here (speaker's choice), but this is left blank in case we need an extra session for the above. Date: Monday, June 3, 2024, **Speaker:**

    Exercise session:. Suggested exercises:. Date: Friday, June 7, 2024
    References:

Possible extra topics:

1. Type B,C,D,E Combinatorics
2. Kazhdan-Lusztig and R-Polynomials
3. Associahedra
4. Cluster Combinatorics
5. Non-crossing partitions

References

[1] A. Bjorner and F. Brenti. *Combinatorics of Coxeter Groups*. Graduate Texts in Mathematics. Springer Berlin Heidelberg, 2006.

[2] N. Bourbaki. *Lie Groups and Lie Algebras: Chapters 4-6*. Number pts. 4-6 in Elements de mathematique [series]. Springer Berlin Heidelberg, 2008.

[3] S. Fomin and N. Reading. Root systems and generalized associahedra, `https://arxiv.org/pdf/math/0505518v3.pdf`. From PCMI 2004.

[4] J.E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1992.

[5] V. Reiner. Warm-up in type a: Noncrossing, nonnesting partitions and associahedra, `https://web.math.ucsb.edu/~jon.mccammond/associahedra/documents/reiner-aim-exercises.pdf`. .

[6] J. Ruiter. Root systems, `https://users.math.msu.edu/users/ruiterj2/math/Documents/Notes%20and%20talks/Root%20systems.pdf`. Lecture Notes.