Affine Deodhar Diagrams and Rational Dyck Paths

GSCC 2025

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(k, n)-**Deograms**

A (k, n)-Deodhar Diagram (**Deogram**) is a filling of boxes of a $k \times (n-k)$ rectangle with crossings, , and elbows, , with

1. strand permutation equal to identity,
2. exactly $n-1$ elbows,
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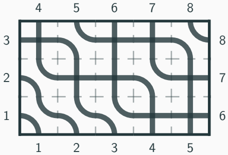
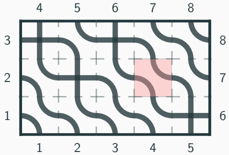


Example

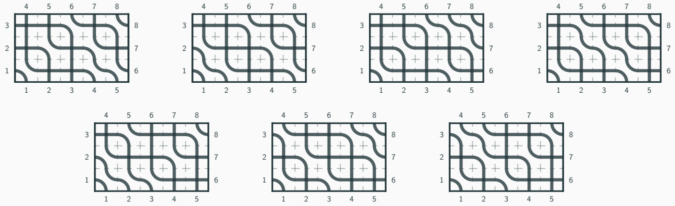


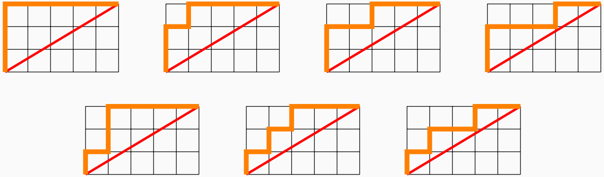
Non-example

Let $\mathrm{Deo}_{k,n}$ denote the set of (k, n)-Deograms.

Overview

Theorem (Galashin-Lam, '22)
For $0 < k < n$ with $\gcd(k, n) = 1$, $\mathrm{Deo}_{k,n}$ and $\mathrm{Dyck}_{k,n}$ are equinumerous.





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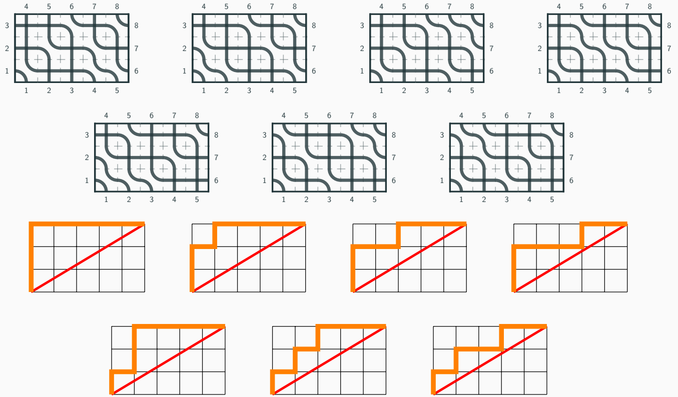
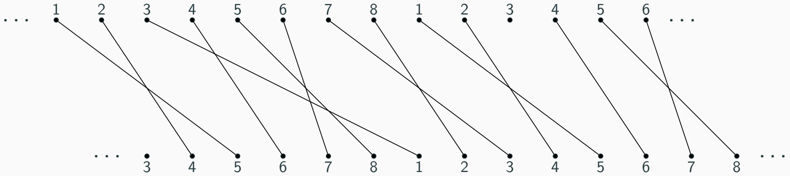


Theorem (M., '25+)
We find a bijection!

Bounded Affine Permutations

A (k, n)-bounded affine permutation is a function $f : \mathbb{Z} \to \mathbb{Z}$ such that:

1. $\sum_{i=1}^{n} f(i) - i = kn$,
2. $i \leq f(i) < i + n$,
3. $f(i + n) = f(i) + n$.



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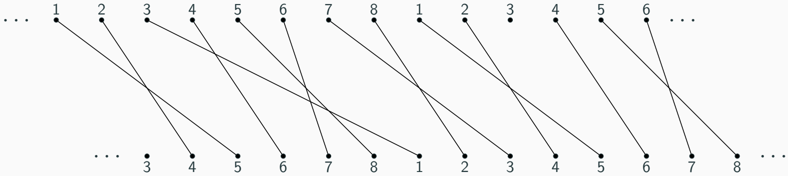
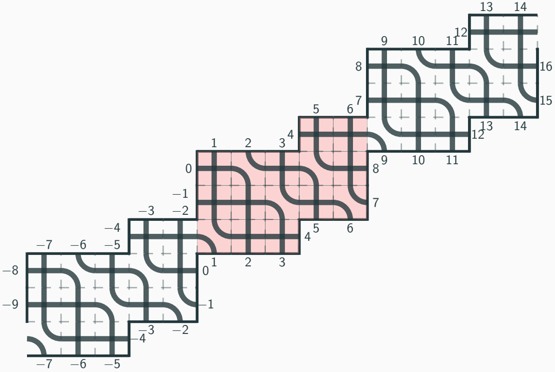


Let $\mathbf{B}_{k,n}$ denote the set of (k,n)-bounded affine permutations.

An f-**affine Deogram** is a *periodic filling* of the space between a path P with k up-steps and $n-k$ right steps and its vertical translate with:

1. Strand permutation equal to $f \in \mathbf{B}_{k,n}$,
2. Exactly $n - (\#\text{cycles of } f)$ elbows (inside a red region),
3. No elbows after an odd number of crossings (from top-left).



We let $\mathrm{AffDeo}_{f,P}$ denote the set of f-affine Deograms under P.

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These are similar to Affine Pipe Dreams introduced by Snider in 2010.

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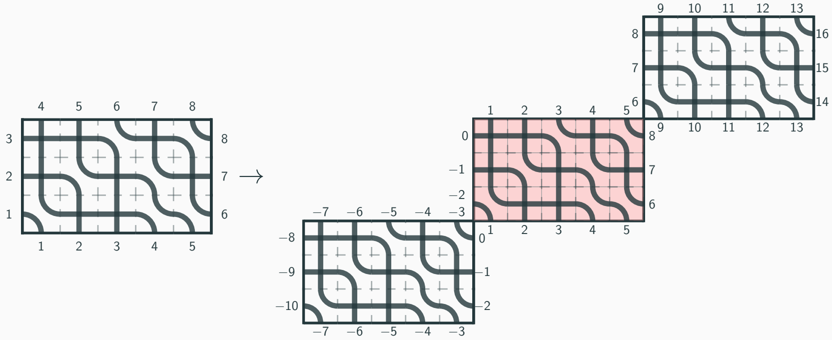
For some paths P, we have a bijection $\mathrm{Deo}_{k,n} \to \mathrm{AffDeo}_{f_{k,n},P}$.



Moves on Affine Deograms

We have 3 moves on f-affine Deograms:

1. Box Addition/Removal
2. Zipper
3. Decoupling

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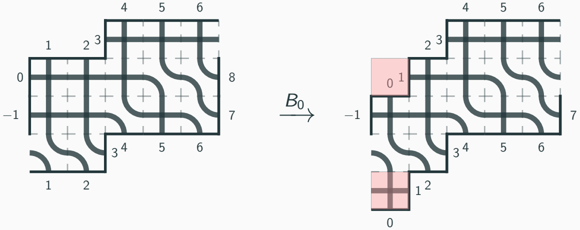
Box Addition/Removal

Motto: We change our path at index i and move the box up/down



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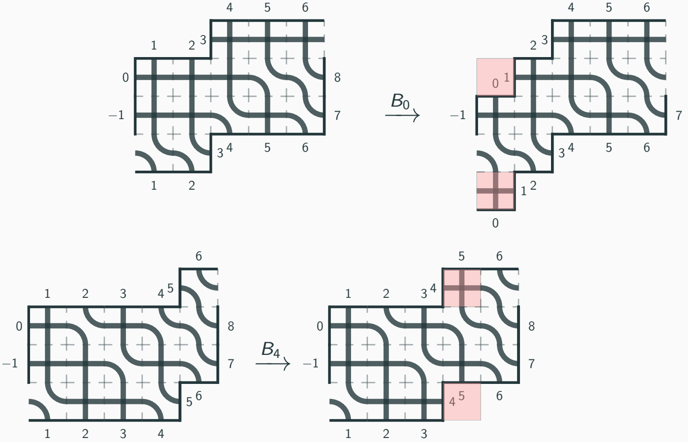

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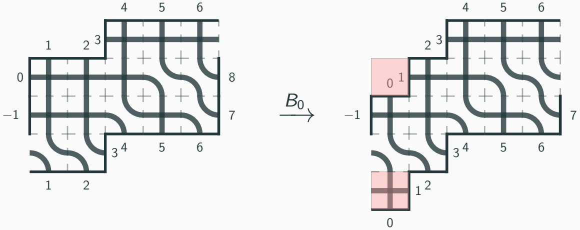

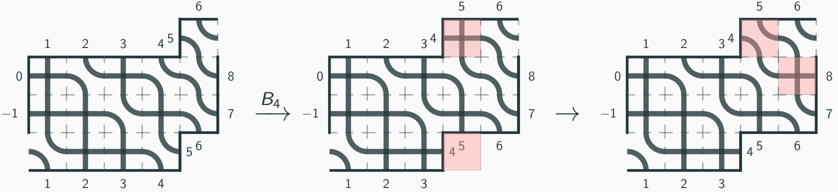

The move B_0 is why we need affine Deograms. It has no simple “lift” to rectangular Deograms.

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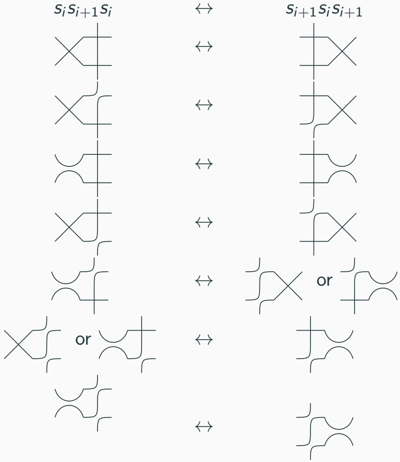
Yang-Baxter Moves



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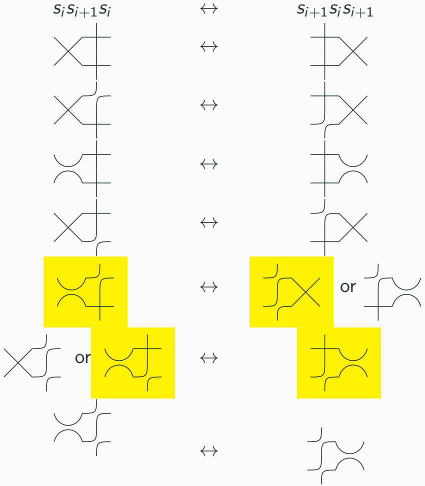

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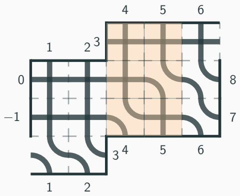
Bijection if we require Condition 3. (No elbow after an odd number of crossings)

Zipper

Motto: We cross wires below and locally apply Yang-Baxter moves until the crossing moves to the top of the path.



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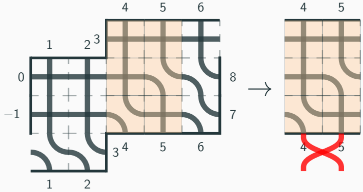

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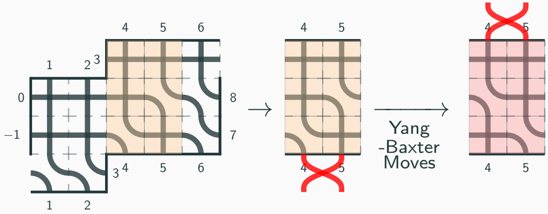

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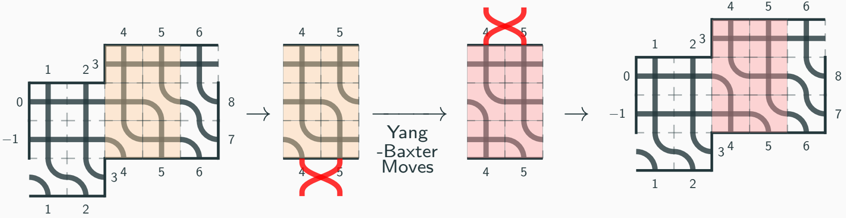

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3. **Decoupling**

Decoupling

Let $f = f_1 f_2 \dots f_r$ be a decomposition of $f \in \mathbf{B}_{k,n}$ into cycles. Then,

$$\# \operatorname{AffDeo}_{f,P} = \prod_{i=1}^{r} \# \operatorname{AffDeo}_{f_i,P_i}.$$

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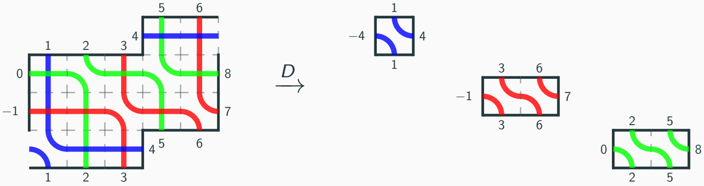
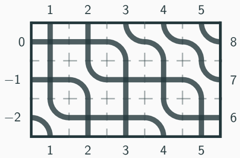


Full Recurrence Example



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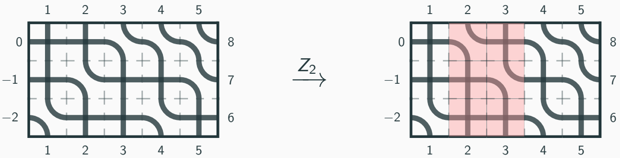

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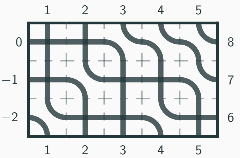

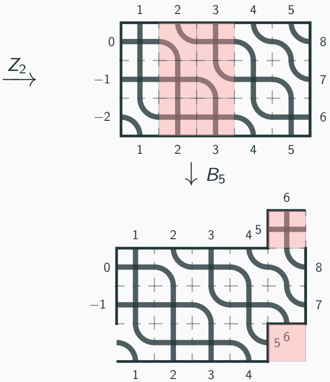

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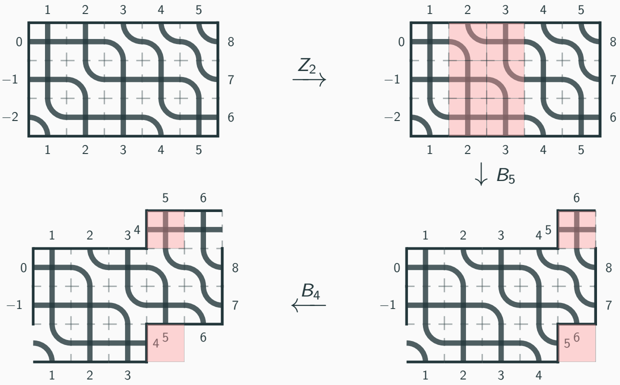

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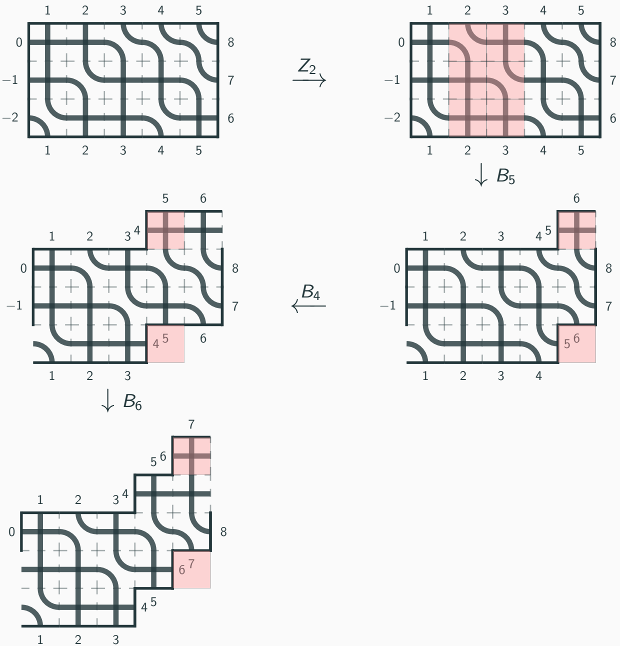

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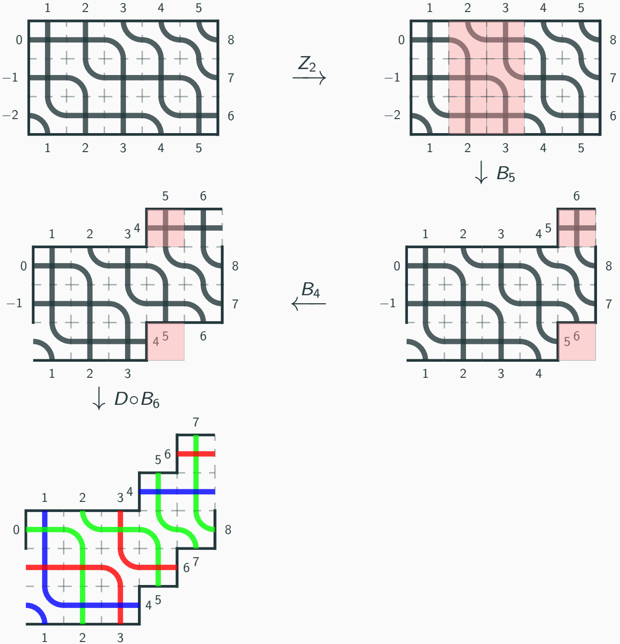

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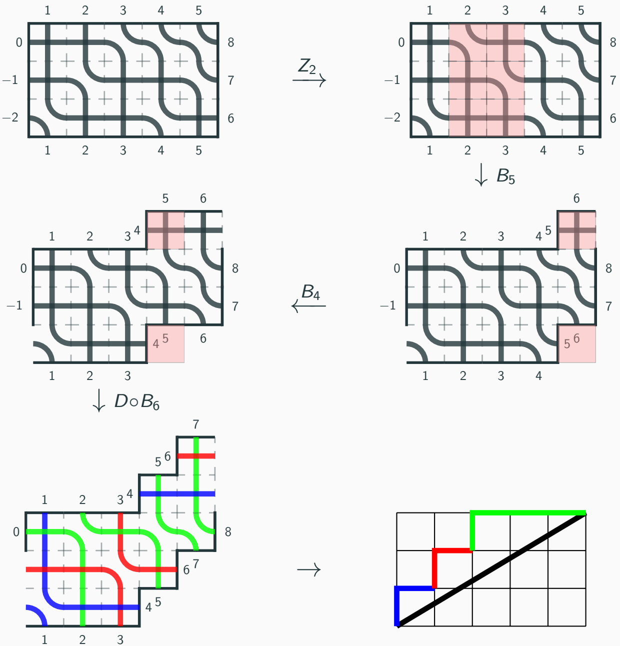

Possible directions

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Can we make this bijection direct?

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So far, yes for:

1. Catalan case, i.e., $(k, k+1)$. (Galashin Lam, '23)
2. 2-row and 2-column case. (M., '25+)

Possible directions

Dyck paths carry a lot of statistics.

$$C_{k,n}(q,t) = \sum_{D \in \mathrm{Dyck}_{k,n}} q^{\mathrm{area}(D)} t^{\mathrm{dinv}(D)}.$$

Question
Can we find statistics on Deograms which makes the bijection statistic-preserving? Can we bijectively prove these statistics are symmetric?

Questions?

Geometric Background

For every $f \in \mathbf{B}_{k,n}$, let $C_f = \chi_T(\Pi_f^\circ)$, the toric-equivariant Euler characteristic of the positroid variety associated to f. Then $C_f = \#\,\mathrm{AffDeo}_{f,P}$, when P is the first element of the Grassmannian necklace for f.

This is also related to

1. Kazhdan-Lusztig R-polynomials,
2. HOMFLY polynomials,
3. Khovanov-Rozansky triply-graded link invariants.