Geometry Qualifying Exam Study Guide

This study guide was created during the summer of 2023 via the collaborative effort of Thomas Martinez, Zach Baugher, Robert Miranda, Olha Shevchenko, Arian Nadjimzadah, John Hopper, Harahm Park, and William Chang as preparation for the Fall 2023 UCLA Geometry Qualifying Exam. We would also like to mention Harris Khan, Jerry Luo, and Sam Qunell for their solutions and notes as this study guide references their writings in many instances. The material is sorted below, where we include many important theorems and (when possible) their proofs followed by selected past qualifying exam problems related to the topics. The end of this PDF also includes solutions from the Fall 2020 to Spring 2023 exams. This is, by no means, a fully exhaustive list of all possible theorems or results that may be useful on the exam; nevertheless, our hope is that this will be a useful aid for those preparing in the future.

Happy studying and good luck – we wish you the very best.

Contents

1 Transversality

We say a map $F : M \to N$ is transverse to a submanifold $S \subset N$ provided

$$T_{F(p)}S + DF(T_pM) = T_{F(p)}N$$

for all $p \in M$ such that $F(p) \in S$. When M is a submanifold of N, F is the inclusion map. The big idea of transversality is that it allows us to obtain a useful generalization of the pre-image theorem. We state the theorem

Theorem 1.1. *If $F : M \to N$ is transverse to a (properly) embedded submanifold $S \subset N$, then $F^{-1}(S) \subset M$ is also a (properly) embedded submanifold. When $F^{-1}(S) \neq \emptyset$, then*

$$\operatorname{codim} F^{-1}(S) = \dim M - \dim F^{-1}(S) = \dim N - \dim S = \operatorname{codim} S.$$

This here is not a proof but a good way to think about how you may arrive at the definition of transversality naturally. This is taken from Guillemin and Pollack.

We have the the solutions of the equation $F(m) = n$ form a smooth manifold, provided n is a regular value of the map $F : M \to N$. When can we say the same if $S \subset N$ is a submanifold and we conside the set of solutions of the relation $F(m) \in S$?

Whether $F^{-1}(S)$ is a manifold is a local matter. This allows us to consider the simpler case when S is a single point. If $n = F(m)$, then we may write S in a neighborhood of n as the zero set of a collection of independent functions $g_1, \ldots, g_\ell$, ℓ being the codimension of S in N. Then, near m, the pre-image $f^{-1}(S)$ is the zero set of the functions $g_1 \circ F, \ldots, g_\ell \circ F$. Let g denote the submersion $(g_1, \ldots, g_\ell)$ defined around n. To the map $g \circ F : W \to \mathbb{R}^\ell$, we may apply the pre-image theorem: $(g \circ F)^{-1}(0)$ is guaranteed to be a manifold when 0 is a regular value of $g \circ F$.

Although the map g is arbitrary, the condition that 0 is a regular value of $g \circ F$ can be reformulated in terms of F and N alone. Since (by chain rule),

$$d(g \circ F)_m = dg_{f(m)} \circ df_m,$$

the linear map $d(g \circ F)_m : T_m(M) \to \mathbb{R}^\ell$ is surjective iff $dg_{f(m)}$ carries the image of dF_m onto $\mathbb{R}^\ell$. But $dg_{f(m)} : T_{f(m)}(N) \to \mathbb{R}^\ell$ is a surjective linear transformation whose kernel is $T_{f(m)}(S)$. Thus, $dg_{f(m)}$ carries a subspace of $T_{f(m)}(N)$ onto $\mathbb{R}^\ell$ precisely if that subspace spans the part of $T_{f(m)}(N)$ which non-zero image. Reformulating this exactly gives that $g \circ F$ is a submersion at $m \in F^{-1}(S)$ if and only if

$$\operatorname{image}(dF_m) + T_{f(m)}(S) = T_{f(m)}(N),$$

which is the definition of transversality.

We now state some important results from Peterson's Manifold Theory notes.

Corollary 1.1.1. *Let $G : M \to N$ be transverse to an embedded submanifold $S \subset N$. A map $F : L \to M$ is transverse to $F^{-1}(S) \subset M$ if and only if $G \circ F$ is transverse to S.*

Let M be a manifold with boundary. If $F : M \to N$, then we denote the restriction to the boundary as $\partial F = F|_{\partial M}$.

Theorem 1.2. *Let $F : M \to N$, where M has boundary. If $S \subset N$ has no boundary and both F and ∂F are transverse to S, then $F^{-1}(S)$ is a submanifold with $\partial(F^{-1}(S)) = F^{-1}(S) \cap \partial M$.*

This result allows us to begin classifying 1-dimensional manifolds.

Theorem 1.3. *A connected one-dimensional manifold is diffeomorphic to either S^1 or $\mathbb{R}$ when it has no boundary and either $[0,1]$ or $[0,\infty)$ when it does.*

Corollary 1.3.1. *A compact manifold with boundary admits no retracts onto the boundary.*

Proof. Consider a map $F : M \to \partial M$ such that $F|_{\partial M} = \mathrm{id}_{\partial M}$. If $p \in \partial M$ is a regular value, then $F^{-1}(p) \subset M$ is a one-dimensional manifold with $\partial(F^{-1}(p)) = F^{-1}(p) \cap \partial M = \{p\}$. Thus, $F^{-1}(p)$ is non-compact and consequently M must also be non-compact. □

Corollary 1.3.2 (Brouwer's Fixed Point Theorem)**.** *Any map on the closed unit ball in Euclidean space has a fixed point.*

1.1 Thom's Transversality Theorem

The previous section dealt with maps $F : M^n \to N$ transverse to $S \subset N$. We now seek to answer the question: given $S \subset N$ and M^n, can we find maps $F : M \to N$ transverse to $S \subset N$?

Lemma 1.4. *Let L be a manifold without boundary and $F : M \times L \to N$. If F and ∂F are transverse to $S \subset N$, then $F_\ell : M \to N$ and $\partial F_\ell : \partial M \to N$ are transverse to S for almost all $\ell \in L$, where $F_\ell(x) = F(x, \ell)$.*

This can be used to prove the Borsuk-Ulam theorem.

Theorem 1.5 (Borsuk-Ulam)**.** *The following statements are equivalent and true*

(a) If $f : S^n \to \mathbb{R}^n$, then there exists $x \in S^n$ such that $f(x) = f(-x)$.

(b) If $f : S^n \to \mathbb{R}^n$ is odd then there exists $x \in S^n$ such that $f(x) = 0$.

(c) There is no odd map $f : S^n \to S^{n-1}$.

Theorem 1.6 (Thom)**.** *Any map $f : M \to N$ is homotopic to a nearby map that is transverse to $S \subset N$. Moreover, if f is a section for $\pi : N \to M$, i.e., $\pi \circ f = id_M$, then the homotopy $H : [0,1] \times M \to N$ can be chosen so that all of the maps $H_t : M \to N$ are sections. Finally, if f is proper, then the homotopy is also proper.*

The proof is long but allows us to set up for Mod 2 Intersection theory. I don't know if it is worth knowing the full proof of Thom, but definitely the next corollaries are important.

Corollary 1.6.1. *Let $F : M \to N$. If $\partial F : \partial M \to N$ is transverse to $S \subset N$, then there is a homotopy $H : [0,1] \times M \to N$ such that $H(t,x) = \partial F(x)$ for all $x \in \partial M$ and $x \mapsto H(1,x)$ is transverse to $S \subset N$.*

In particular, if two maps are homotopic and transverse to S, there there exists a homotopy between the maps that is also transverse to S.

Corollary 1.6.2. *Any manifold admits a vector field that is transverse to the zero section $p \mapsto 0_p \in T_pM$.*

Corollary 1.6.3. *Any map $F : M \to M$ is homotopic to a map $G : M \to M$ such that $(id_M, G) : M \to M \times M$ where $x \mapsto (x, G(x))$ is transverse to the diagonal $\Delta = \{(p,p) | \, p \in M\}$.*

Proof. Just use that (id_M, F) is a section of the projection $\pi_1 : M \times M \to M$ on to the first coordinate. □

1.2 Problems

I could not find this definition in Petersen's note but here is something that could appear.

Two manifolds $K, L \subset M$ are said to intersect transversely if for every point $p \in K \cap L$, we have

$$T_pK + T_pL = T_pM.$$

This means that K and L are transverse if and only if the inclusion map $\iota : K \hookrightarrow M$ is transverse to L in the usual sense, or vice versa for $\iota' : L \hookrightarrow M$. The idea is that if this is true, then $K \cap L$ is also a submanifold of M, and $\operatorname{codim} K \cap L = \operatorname{codim} K + \operatorname{codim} L$, by applying Theorem 1.1.

Spring 2016, 2. Let X and Y be submanifolds of $\mathbb{R}^n$. Prove that, for almost every $a \in \mathbb{R}^n$, the translate $X + a$ intersects Y transversely.

Solution. Consider the map $F : X \times \mathbb{R}^n \to \mathbb{R}^n$ where $(x, a) \mapsto x + a$. We claim F is transversal to Y. If the image of F and Y do not intersect, then they are trivially transversal. Otherwise, suppose we have $F(x, a) \in Y$. then we note $dF_{(x,a)} : T_xX \times T_a\mathbb{R}^n \to T_y\mathbb{R}^n$ can be written as a block matrix with identity on the bottom, since F is the identity on the second entry. Thus,

$$dF_{(x,a)}(T_xX \times T_a\mathbb{R}^n) = T_y\mathbb{R}^n.$$

Thus, F is transverse to Y. By an above lemma, this implies the map $F_a : X \to \mathbb{R}^n$ where $x \mapsto x + a$, or equivalently the inclusion map $F_a : X + a \hookrightarrow \mathbb{R}^n$ where $x + a \mapsto x + a$ is transverse to Y for almost all $a \in \mathbb{R}^n$. □

Fall 2016, 2 Let $M \subset \mathbb{R}^N$ be a smooth k-dimensional submanifold. Prove that M can be immersed into $\mathbb{R}^{2k}$.

WLOG we may assume $N > 2k$, as if not we have an immersion $M \to \mathbb{R}^{2k}$ via the inclusion $M \to \mathbb{R}^N \hookrightarrow \mathbb{R}^{2k}$.

We show this inductively. Let $f : M \to \mathbb{R}^m$ be an immersion where $m > 2k$. Define $g : TM \to \mathbb{R}^m$ where $g(x, v) = df_x(v)$. Recall that $\dim TM = 2k$. Then since $m > 2k$ we know every point in TM is a critical point of g. By applying Sard's theorem, we know that the image of g is a set of measure zero in $\mathbb{R}^m$. Therefore, we can pick an $a \in \mathbb{R}^m$ such that $a \notin g(TM)$ and $a \neq 0$. Let π be the projection of $\mathbb{R}^m$ onto the orthogonal complement of a, H_a. we show the composition $\pi \circ f : M^k \to H_a$ is an immersion.

Suppose for contradiction that $v \neq 0$, and $v \in T_xM$ such that $d(\pi \circ f)_x(v) = 0$. Note that since π is linear, $d(\pi \circ f)_x = \pi \circ df_x$ by the chain rule. So $\pi \circ df_x(v) = 0$ which implies $df_x(v) = ta$ for some $t \in \mathbb{R}$. If $t = 0$, then $df_x(v) = 0$, which is impossible as f is an immersion and $v \neq 0$. So $t \neq 0$. Therefore, we must have $g(x, \frac{1}{t}) = df_x(\frac{1}{t}) = a$ which is a contradiction since $a \notin g(TM)$. So $d(\pi \circ f)$ is injective implying $\pi \circ f$ creates a immersion into an $m - 1$ dimensional subspace of $\mathbb{R}^m$, which is isomorphic to $\mathbb{R}^{m-1}$. ∎

2 Mod 2 Intersection

2.1 Definitions and First Results

Notes on this section are drawn from Petersen's notes and from Guillemin & Pollack.

The setup for the section is as follows: we have a smooth map $F : M \to N$ for M compact and N connected. Let $S \subset N$ be a closed submanifold, assume F is transverse to S and that $F(\partial M) \cap S = \varnothing$. Then, if $\dim M + \dim S = \dim N$, the preimage $F^{-1}(S) \subset M$ is a compact 0-manifold, so a finite collection of points of $M \backslash \partial M$.

Remark 2.1. *We can get away with M non-compact as long as F is proper. In this case, we can take the required homotopies in the proofs through proper maps.*

Definition 2.1. *In the setup of the previous paragraph, the* mod 2 intersection number $I_2(F,S)$ *is the value* $\#F^{-1}(S) \pmod 2$.

One important advantage of working mod 2 is that the intersection mod 2 is well-defined up to homotopy, unlike the honest count $\#F^{-1}(S)$.

Theorem 2.1. *Let $F_0, F_1 : M^m \to N^n$ be homotopic maps transverse to $S^{n-m} \subset N$, such that when $\partial M \neq \varnothing$ we have $\partial F_0 = \partial F_1$, neither boundary map intersects S, and the homotopy is fixed on ∂M. Then $I_2(F_0,S) = I_2(F_1,S)$.*

Proof sketch. In the boundaryless case, consequences of Thom's transversality theorem produce a homotopy $H : [0,1] \times M \to N$ transverse to S, so that $H(0,x) = F_0(x)$ and $H(1,x) = F_1(x)$. Then $H^{-1}(S)$ is a compact one-manifold with boundary, so that $\partial H^{-1}(S)$ has an even number of points. But

$$\partial H^{-1}(S) = H^{-1}(S) \cap \{0,1\} \times \operatorname{int} M = \{0\} \times F_0^{-1}(S) \cup \{1\} \times F_1^{-1}(S),$$

so $\#\partial H^{-1}(S) = \#F_0^{-1}(S) + \#F_1^{-1}(S)$ is even, whence $F_0^{-1}(S)$ and $F_1^{-1}(S)$ have the same parity and so $I_2(F_0,S) = I_2(F_1,S)$.

When M has boundary, the product $[0,1] \times M$ is no longer a manifold with boundary, but Petersen assures us we can still get a homotopy as above since $\partial F_0 = \partial F_1$ don't intersect S. ☐

We can now define mod 2 intersection for maps F not necessarily transverse to S:

Definition 2.2. *In our setup with $F : M \to N$ not necessarily transverse to $S \subset N$, we define $I_2(F,S)$ to be $I_2(G,S)$ for any map $G : M \to N$ transverse to S and homotopic to F. If M has boundary we require ∂F not to intersect S and the homotopy to restrict to ∂F on ∂M for all times $t \in [0,1]$.*

Example 2.1. *The identity map $id_M : M \to M$ is transverse to $\{x\} \subset M$ with $\#id_M^{-1}(\{x\}) = 1$, so $I_2(id_M, \{x\}) = 1$.*

Theorem 2.2. *Let B^{m+1} be a compact manifold with boundary $\partial B = M^m$ and $f : M^m \to N^n$ with $S^{n-m} \subset N^n$ a closed submanifold. If $f = \partial F$, where $F : B \to N$, then $I_2(f,S) = 0$.*

Proof sketch. Use Thom's transversality theorem to obtain a map $G : B \to N$ such that G and ∂G are both transverse to S. Now $G^{-1}(S)$ is a compact one-manifold, so $\partial(G^{-1}(S)) = (\partial G)^{-1}(S)$ has an even number of boundary points. Then $0 = I_2(\partial G, S) = I_2(f,S)$. ☐

Remark 2.2. *This theorem recovers the result that a compact manifold never retracts onto its boundary: any such $r : M \to \partial M$ has $id_{\partial M} = \partial r$, so the theorem gives $I_2(\partial r, \{x\}) = 0$ for any $\{x\} \in \partial M$, contradicting $I_2(id_{\partial M}, \{x\}) = 1$.*

Definition 2.3. *The* mod 2 Euler characteristic *of a manifold M is $\chi_2(M) = I_2(X, M_0)$, where X is a vector field on M and M_0 the zero section. This is well-defined since every manifold admits a vector field transverse to the zero section and all vector fields on M are homotopy equivalent.*

Definition 2.4. *The* mod 2 Lefschetz number *of $F : M \to M$ is defined by $L_2(F) := I_2((id_M, F), \Delta)$, where $\Delta \subset M \times M$ is the diagonal. This is well-defined since such an F is always homotopic to a G for which (id_M, G) is transverse to Δ.*

Proposition 2.1. *We have $\chi_2(M) = L_2(id_M)$.*

2.2 The Winding Number and the Jordan-Brouwer Separation Theorem

The approach to this section is taken from section 2.5 in Guillemin & Pollack. We give statements here; the proofs are exercises in G&P.

We require a couple more definitions.

Definition 2.5. *Let $F : M \to N$ be a smooth map with M compact, N connected, and* $\dim M = \dim N$. *Then the* mod 2 degree of *F is defined by* $\deg_2(F) = I_2(F, \{p\})$ *for any $p \in N$.*

Note that there is a concrete description of $\deg_2 F$: it is the number $\#F^{-1}(q)$ of preimages of a regular value $q \in N$, modulo 2. Moreover, since mod 2 degree is defined as an intersection number, the results from the previous section apply; in particular, mod 2 degree is homotopy-invariant, and the mod 2 degree of a map $F : M \to N$ equal to ∂G for some $G : B \to Y$ with $\partial B = M$ is zero.

To prove Jordan-Brouwer, we track one final invariant:

Definition 2.6. *Let $F : M^n \to \mathbb{R}^{n+1}$ be a smooth map with M compact and connected. The* mod 2 winding number *of F with respect to a point $z \notin F(M)$ is given by*

$$W_2(f, z) = \deg\left(\frac{F(x) - z}{|F(x) - z|} : M \to S^{n-1}\right).$$

Thus the mod 2 winding number counts whether $F(x) - z$ points in some generic direction an even or odd number of times. We have the following theorem on winding numbers.

Theorem 2.3. *Let X be a compact, connected manifold of dimension $n - 1$ and suppose we are given a smooth map $f : X \to \mathbb{R}^n$. Suppose also that $X = \partial D$ for D a compact manifold with boundary, and that we have a smooth map $F : D \to \mathbb{R}^n$ extending f (i.e. $f = \partial F$). If z is a regular value of F which is not in the image of f, then $F^{-1}(z)$ is finite and $W_2(f, z) = \#F^{-1}(z) \pmod 2$.*

Theorem 2.4 (Jordan-Brouwer Separation Theorem)**.** *Let $X \subset \mathbb{R}^n$ be a compact, connected hypersurface (i.e. X has codimension 1). Then $\mathbb{R}^n \backslash X$ consists of two connected open sets D_0, D_1, and we can choose the labeling so that $\overline{D_1}$ is a compact manifold with boundary X.*

For a proof see Petersen's notes. Part of the proof is outlined in the solution to Fall 2014, problem 2 below.

2.3 Problems

Spring 2018, 3 For $n \geqslant 1$, consider the subset $X \subset \mathbb{CP}^{2n}$ given by

$$X = \{[z_0 : z_1 : \cdots : z_{2n}] \in \mathbb{CP}^{2n} \mid z_{n+1} = \cdots = z_{2n} = 0\}.$$

(a) Show that X is a smooth submanifold.
(b) Calculate the mod 2 intersection number of X with itself.

Solution. (a) Define a map $F : \mathbb{CP}^n \to \mathbb{CP}^{2n}$ by $[z_0 : \cdots : z_n] \mapsto [z_0 : \cdots : z_n : 0 : \cdots : 0]$. We will show that F is a smooth embedding with image X, which suffices to prove that X is a smooth submanifold.

To see that F is smooth, note that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^{n+1}\backslash\{0\} & \xrightarrow{\iota} & \mathbb{C}^{2n+1}\backslash\{0\} \\ \pi\downarrow & & \downarrow\pi \\ \mathbb{CP}^n & \xrightarrow[F]{} & \mathbb{CP}^{2n} \end{array}$$

where both maps labelled π are the projections, both surjective smooth submersions by the construction of complex projective space as a manifold, and ι is the inclusion into the first $n+1$ coordinates of $\mathbb{C}^{2n} + 1\backslash\{0\}$, which is clearly smooth. We can then pass to the quotients to determine that F is smooth.

It remains to show F is an embedding with image X. That F has image X is clear. To show F is an embedding, recall that embeddings are precisely proper injective immersions. We have that F is proper because $\mathbb{CP}^n$ is compact, and F is seen to be injective by construction. To show F is an immersion, one can write out the transition maps in real coordinates and check that the Jacobian matrices have constant rank n. If fact, when you work in local coordinates (WLOG, suppose $z_0 \neq 0$) then the map becomes a canonical projection, ie $(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}) \to (\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}, 0, \ldots, 0)$.

To calculate the mod 2 intersection number of X with itself, we produce a map $G : \mathbb{CP}^n \to \mathbb{CP}^{2n}$ which is transverse to X and homotopic to F, then calculate the mod 2 intersection number of G with X. The map G we will use sends $[z_0 : \ldots z_n]$ to $[0 : 0 : \cdots : z_n : \cdots : z_0]$, where z_n occurs in position $n+1$. Define a map $H : \mathbb{CP}^n \times [0,1] \to \mathbb{CP}^{2n}$ by sending $([z_0 : \cdots : z_n], t)$ to $[tz_0 : \cdots : tz_{n-1} : z_n : (1-t)z_{n-1} : \cdots : (1-t)z_0]$. Then $H(x,0) = G(x)$ and $H(x,1) = G(x)$, and H is smooth by another passing-to-quotients argument (noting that it is constant on projective equivalence classes). So F and G are homotopic, and to show G is transverse to X we need to show their tangent spaces add to $T_p\mathbb{CP}^{2n}$ at every point of $X \cap G(\mathbb{CP}^n)$. The only point of this intersection is $p = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$, where 1 is in position $n+1$. Now calculate that $T_pX \subset T_p\mathbb{CP}^{2n}$ consists of vectors whose last $2n$ real coordinates are zero, while $dG(T_{[0:\cdots:0:1]}\mathbb{CP}^n)$ consists of those vectors whose first $2n$ real coordinates are zero. It follows that G is transverse to X, and since the intersection $X \cap G(\mathbb{CP}^n)$ has one point, the mod 2 intersection number is 1. $\square$

Fall 2014, 2 Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension m.

(a) Show that $\mathbb{R}^n \backslash M^m$ is connected when $m \leqslant n-2$.

(b) When $m = n-1$, show that $\mathbb{R}^n \backslash M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

Solution. (a) We show that $\mathbb{R}^n \backslash M^m$ is path connected when $\operatorname{codim} M \geqslant 2$. Pick $p, q \in \mathbb{R}^n \backslash M^m$, and let $\gamma : [0,1] \to \mathbb{R}^n$ be the line segment from p to q in $\mathbb{R}^n$. By Thom's transversality theorem, there is a smooth curve $\eta : [0,1] \to \mathbb{R}^n$ which is homotopic to γ in a manner fixing the points p, q, with η transverse to M. Since M has codimension at least 2 and the image of η is one-dimensional, it is not possible for η to be transverse to M unless η and M do not intersect. Thus η is a path from p to q not meeting M, and $\mathbb{R}^n \backslash M^m$ is connected.

(b) First we prove the given claim about intersection numbers of maps $f : S^1 \to \mathbb{R}^n$. Assume without loss of generality that f is transverse to M, let X denote the image of f, and pick distinct points $p, q \in X \backslash M$. These two points give two distinct paths γ_1, γ_2 from p to q along X. Since $\mathbb{R}^n$ is simply connected, the paths γ_1 and γ_2 are homotopic by a homotopy fixing p, q. Moreover γ_1, γ_2 are both transverse to M since X is. So $I_2(\gamma_1, M) = I_2(\gamma_2, M)$, and

$$I_2(f, M) = I_2(\gamma_1, M) + I_2(\gamma_2, M) = 2I_2(\gamma_1, M) = 0.$$

Now we use this fact to prove $\mathbb{R}^n \backslash M$ is disconnected. Suppose that $\mathbb{R}^n \backslash M^m$ is connected. As M is a closed submanifold of dimension m, given some $m \in M$, there exists some open subset $m \in U \subset \mathbb{R}^n$ and a chart such that $U \cap M = \{(x_1, \ldots, x_m, 0)\}$. Then, there exists some $\epsilon > 0$ such that $(x_1, \ldots, x_m, \epsilon), (x_1, \ldots, x_m, -\epsilon) \in U$ for some choice of $x_1, \ldots, x_m$. Denote these points as p and q respectively. Considering the path $\gamma_0 : [0,1] \to \mathbb{R}^n$ where

$$\gamma_0(t) = (x_1, \ldots, x_m, (1-2t)\epsilon),$$

we note that $I_2(\gamma_0, M) = 1$. However, as $\mathbb{R}^n \backslash M^m$ is connected, there exists another path γ_1 connecting p and q entirely contained in $\mathbb{R}^n \backslash M$. Thus, $I_2(\gamma_1, M) = 0$. However, then the loop $\gamma = \gamma_0 \circ \gamma_1$ has intersection $I_2(\gamma, M) = I_2(\gamma_0, M) + I_2(\gamma_1, M) = 1$, which is a contradiction, as desired.

□

3 Lefschetz Fixed Point Formula

In this section, M is a compact orientable manifold. We define the graph of $f : M \to M$ to be $\Gamma(f) = \{(x, f(x)) \in M \times M\}$, and the diagonal to be $\Delta = \Gamma(\mathrm{id}_M) = \{(x, x) \in M \times M\}$.

We define a map f to be a Lefschetz map if $\Gamma(f) \pitchfork \Delta$. Via Thom (or a related corollary), we obtain the following

Lemma 3.1. *Every map $f : X \to X$ is homotopic to a Lefschetz map.*

What does it mean for a map to be Lefschetz? Recalling the definition of transversality, we need only consider the fixed points of f. The tangent space of $\Gamma(f)$ in $T_xM \times T_xM$ is simply the graph of the map $df_x : T_xM \to T_xM$ and the tangent space of the diagonal Δ is the diagonal Δ_x of $T_xM \times T_xM$. Thus, $\Gamma(f) \pitchfork \Delta$ at (x, x) if and only if

$$\Gamma(df_x) + \Delta_x = T_xM \times T_xM.$$

As $\Gamma(df_x)$ and Δ_x are vector subspaces of $T_xM \times T_xM$, both of dimension $\dim M$, they fill out everything precisely if their intersection is 0. But $\Gamma(df_x) \cap \Delta_x = 0$ means that df_x has no nonzero fixed point (or in the language of linear algebra, df_x has no eigenvector of eigenvalue $+1$).

We call the fixed point x a Lefschetz fixed point of f if df_x has no nonzero fixed point. f is then a Lefschetz map if and only if all fixed points are Lefschetz fixed points.

If x is a Lefschetz fixed point, we denote the orientation number of (x, x) in the intersection of $\Delta \cap \Gamma(f)$ by $L_x(f)$ (called the local Lefschetz number of f at x). Thus, for Lefschetz maps,

$$L(f) = \sum_{f(x)=x} L_x(f).$$

To find $L_x(f)$, we first note that x being a Lefschetz fixed point is equivalent to $df_x - I$ being an isomorphism of T_xM. Thus, we get the following

Lemma 3.2. *The local Lefschetz number $L_x(f)$ at a Lefschetz fixed point x of f is $+1$ if the isomorphism $df_x - I$ preserves orientation on T_xM and -1 if it reverses orientation. That is, the sign of $L_x(f)$ equals the sign of the determinant of $df_x - I$.*

This result is one in which you should know. It is easy to remember and a common method of proving a map has a fixed point.

Theorem 3.3. *Let $f : M \to M$ be a smooth map on a compact orientable manifold. If $L(f) \neq 0$ then f has a fixed point.*

Proof. We prove the contrapositive. If f has no fixed points then Δ and $\Gamma(f)$ are trivially transversal as they are disjoint. Thus, $L(f) = I(\Gamma(f), \Delta) = 0$. ☐

How can we calculate Lefschetz numbers? In some instances (See Spring 2013 #6), we can calculate it outright. However in many instances, we use the formula below.

Theorem 3.4 (Lefschetz Trace Formula)**.** *If $F : M \to M$, then*

$$L(F) = I(\Gamma(F), \Delta) = \sum (-1)^p tr(F^* : H^p(M; \mathbb{Q}) \to H^p(M; \mathbb{Q})).$$

Note: there are many different statements of this formula, sometimes defined on homology. Here is another formulation. It is almost important to note that this is always well-defined when either $H^p(M; R)$ is free. When R is a field, this is guaranteed, but we may calculate this in integral coefficients in the case where all homology groups are free (e.g. $\mathbb{C}P^n$).

Theorem 3.5 (Lefschetz Trace Formula, v2)**.** *If $F : M \to M$, then*

$$L(F) = I(\Gamma(F), \Delta) = \sum (-1)^p tr(F_* : H_p(M; \mathbb{Q}) \to H_p(M; \mathbb{Q})).$$

This may be hard to calculate in general, but it is easy when $p = 0$ or $p = \dim M$ and M is compact, connected, and oriented. In which case

$$\mathrm{tr}(F^* : H^0(M; \mathbb{Q}) \to H^0(M; \mathbb{Q})) = 1,$$

$$\mathrm{tr}(F^* : H^n(M; \mathbb{Q}) \to H^n(M; \mathbb{Q})) = \deg F.$$

This formula implies Brouwer's Fixed point theorem almost instantly.

3.1 Problems

Spring 2015, 4 Consider a smooth map $f : \mathbb{R}P^n \to \mathbb{R}P^n$.
(a) When n is even show that F has a fixed point.
(b) When n is odd, give an example where F does not have a fixed point.

(a) To show F has a fixed point, we show that $L(f) \neq 0$. We show this via Lefschetz's trace formula. Note that, for n even,

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2\mathbb{Z} & 1 \leqslant k < n \text{ and } k \text{ odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Thus, by the universal coefficient theorem, we have that

$$H^k(\mathbb{R}P^n; \mathbb{Q}) = \operatorname{Ext}(H_{k-1}(\mathbb{R}P^n), \mathbb{Q}) \oplus \operatorname{Hom}_{\mathbb{Z}}(H_k(\mathbb{R}P^n), \mathbb{Q}).$$

However, we note that

$$\operatorname{Ext}(\mathbb{Z}, \mathbb{Q}) = 0$$

as $\mathbb{Z}$ is free and

$$\operatorname{Ext}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Q}/2\mathbb{Q} \cong \mathbb{Q}/\mathbb{Q} \cong 0$$

and $2\mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Q}$ is a field. Additionally, we note that

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Q}) = 0$$

as we require $0 \mapsto 0$ and $1 \mapsto b$ implies $b + b = 0$, thus we need $b = 0$. Finally, we note that

$$\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}) \cong \mathbb{Q}$$

as each map is uniquely determined by the image of 1, as $\phi(m) = m\phi(1)$ for every $m \in \mathbb{Z}$. Thus, $H^*(\mathbb{R}P^n; \mathbb{Q}) = \mathbb{Q}_{(0)}$. So, any map $F : \mathbb{R}P^n \to \mathbb{R}P^n$, induces the identity map $F^* : H^0(\mathbb{R}P^n; \mathbb{Q}) \to H^0(\mathbb{R}P^n; \mathbb{Q})$. Thus, by the Lefschetz trace formula,

$$L(f) = \operatorname{tr}(F^* : H^0(\mathbb{R}P^n; \mathbb{Q}) \to H^0(\mathbb{R}P^n; \mathbb{Q})) = 1 \neq 0,$$

as desired.

(b) Consider the map $F : \mathbb{R}P^n \to \mathbb{R}P^n$ where

$$[a_0 : a_1 : \cdots : a_{n-1} : a_n] \mapsto [-a_1 : a_0 : \cdots : -a_n : a_{n-1}].$$

This map has no fixed points as $F(x)$ is normal to x in local coordinates, and at least one $a_i \neq 0$.

■

Fall 2015, 8 Show that $\mathbb{C}P^n$ is not a covering space for any manifold other than itself when n is even.

Note that, for n even,

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & 0 \leqslant k \leqslant 2n,\ k \text{ even}, \\ 0 & \text{otherwise.} \end{cases}$$

We claim that every diffeomorphism $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point. Supposing this claim is true, if $p : \mathbb{C}P^2 \to X$ was a covering map, then every element of $\pi_1(X)$ acts by diffeomorphisms, and thus has a fixed point. However, the only element of the deck group which fixes any element is the identity element. This implies $\pi_1(X)$ is trivial, thus $X = \mathbb{C}P^n$.

Let us now prove our claim. Let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ a diffeomorphism be given. Then, by the Lefschetz trace formula,

$$L(f) = \sum_{i=0}^{n} \operatorname{tr}(f_* : \mathbb{Z} \to \mathbb{Z}).$$

As f is a diffeomorphism, the induced map on homology is an isomorphism and thus has trace ± 1, thus, $L(f) = \sum \pm 1$, with an odd number of summands. Particularly $L(f) \neq 0$, and thus f has a fixed point. ∎

4 Euler Characteristic

4.1 Definitions

Definition 4.1. *$\chi(M) = I(X, M_0) = I(M, M)$.*

Alternatively, we can define it as

$$\chi(M) = \sum_n (-1)^n c_n,$$

where c_n is the number of n-cells in X.

This definition says that the Euler characteristic is the intersection number of any vector field X and M_0. Recall that every smooth vector field is homotopic (this can be done by a straight line homotopy $H(t, p) = tX_1(p) + (1 - t)X_0(p)$. We can note that in order to compute this intersection number we only need a orientation of $T(TM)$ around the sub-manifold M_0 (which is the image of the zero section). We can note that the tangent space at a point $0_p \in TM$ looks like $TM \times TM$ in local coordinates (we can let the first be the M factor and the second represent the TM factor). This space has a natural orientation which, given any basis of T_pM, $e_1, \ldots, e_m$ then $(e_1, 0), \ldots, (e_m, 0), (0, e_1), \ldots, (0, e_m)$ is well defined and gives the same orientation.

Theorem 4.1. *The Euler Characteristic can be also computed as*

$$\chi(X) = \sum_n (-1)^n \operatorname{rank} H_n(X) = \sum_n (-1)^n \operatorname{rank} H_n(X; F)$$

for any field F (eg. $\mathbb{Q}, \mathbb{R}$).

This fact comes from a pretty straightforward algebraic computation (since c_n is the rank of C_n, we can just write short exact sequences and compare the ranks of the terms). But most importantly, this shows that Euler characteristic is a homotopy invariant.

Proposition 4.1. *For a closed oriented manifold M, $L(id_M) = \chi(M)$*

Proof. Proof Sketch: What is the idea of this well lets just write out $L(id_M) = I((Id_m, Id_m), \Delta) = I(\Delta, \Delta)$ which is the intersection number of the diagonal with itself. We can now 'cock your head sideways' and note that the diagonal can be identified with the manifold M where TM is identified by $N(\Delta) = \{(-v, v), v \in TM\}$. What we need to check is that this identification preserves orientations which is does, we can see this by preforming elementary row operations that don't change the sign of the determinate on $(e_1, e_1), \ldots, (e_m, e_m), (-e_1, e_1), \ldots, (-e_m, e_m)$ to get $(e_1, 0), \cdots (e_m, 0), (0, e_1), \ldots, (0, e_m)$ which is the standard basis. With this identification, we can note that Δ becomes the zero section and we get $I(M_0, M_0) = \chi(M)$ where X is just the zero section which is a vector field. □

4.2 Useful facts

Corollary 4.1.1. *For a closed manifold M of odd dimension m, we have $L(id_M) = \chi(M) = 0$.*

This comes from the fact that $(e_1, e_1), \ldots, (e_m, e_m), (e_1, -e_1), \ldots, (e_m, -e_m)$ has the opposite orientation of that we gave $N(\Delta)$, and we could just as easily identified TM by $v \to (v, -v)$ and $(-v, v)$. Another way of seeing this: via Poincaré duality (in $\mathbb{Z}/2$ coefficients as every manifold is $\mathbb{Z}/2$-orientable) since we know that M is a closed manifold there is an isomorphism between $H_k(M)$ and $H_{m-k}(M)$ and we can note that when m is odd k and $m-k$ have opposite parity so

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \mathrm{Rank}(H_i(M)) = \sum_{i=0}^{(m-1)/2} (-1)^i \mathrm{Rank}(H_i(M)) + (-1)^{m-i} \mathrm{Rank}(H_{m-i}(M)) = 0.$$

Theorem 4.2 (Also S16# 4 and S22 # 9!). *Suppose M is an odd-dimensional compact manifold. Then*

$$\chi(\partial M) = 2\chi(M).$$

Proof. Take two copies M_1 and M_2 and glue them along the boundary to get the manifold $N = M_1 \cup_{\partial M} M_2$. Notice that N satisfies the conditions of the 4.1.1, so $\chi(N) = 0$. Now it is enough to show that $\chi(N) = \chi(M_1) + \chi(M_2) - \chi(\partial M) = 2\chi(M) - \chi(\partial M)$.

Notice that this is one of the properties in 4.3, but we will prove it more carefuly. For that, take neighborhoods ϵ_1 and ϵ_2 of ∂M in M_1 and M_2 that can be deformation retracted onto ∂M. Now, apply the Mayer-Vietoris sequence for $N = (M_1 + \epsilon_2) \cup (M_2 + \epsilon_2)$:

$$\ldots \to H_{n+1}(N) \to H_n(\partial M) \to H_n(M_1) \oplus H_n(M_2) \to$$
$$\to H_n(N) \to H_{n-1}(\partial M) \to H_{n-1}(M_1) \oplus H_{n-1}(M_2) \to \ldots$$

Now it is only left to apply the following fact:

Lemma 4.3. *In every finite exact sequence, the alternating sum of the ranks of the groups is equal to 0.*

If we combine this with 4.1, we get the desired equality. □

Theorem 4.4 (Poincaré - Hopf). *Let X be a vector field on the compact oriented manifold M with isolated zeros x_i. Suppose also that X is pointing in the normal direction along the boundary. Then*

$$\chi(M) = \sum_i \mathrm{index}_{x_i}(X),$$

where $\text{index}_{x_i}(X)$ *is the degree of the map* $u : \partial D \to S^{n-1}$ *defined by* $u(z) = X(z)/\|X(z)\|$ *(D is a neighborhood of* x_i *in which* x_i *is the only zero).*

This means that to calculate the Euler characteristic $\chi(M)$ of M, we can construct any smooth vector field on this manifold and compute the sum of the indices of this vector field.

4.3 Computing Euler Characteristic

Proposition 4.2. *The Euler characteristic of the common spaces are:*

- $\chi(S^n) = 1 + (-1)^n$.
- $\chi(\mathbb{R}P^n) = \frac{1+(-1)^n}{2}$.
- $\chi(\mathbb{C}P^n) = n + 1$.
- $\chi(M_g) = 2 - 2g$.
- $\chi(N_g) = 2 - g$.
- $\chi(\bigvee_m S^1) = 1 - m$.
- $\chi(T^2) = 0$.

There are also a couple of properties that can help compute the Euler characteristic:

Proposition 4.3. *(a)* $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

(b) If $p : \widetilde{X} \to X$ *is an n-sheeted cover space, then*

$$\chi(\widetilde{X}) = n\chi(X).$$

(c) If $X = A \cup B$, *where A and B are subcomplexes, then*

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B).$$

(d) In particular, $\chi(X \vee Y) = \chi(X) + \chi(Y) - 1$.

Proof. Might be good to know, since some of these are just exercises in Hatcher. But this should just follow from the definition with cell complexes? □

4.4 Problems

Fall 2012, 8 Show that there is no compact 3-manifold M whose boundary is $\mathbb{R}P^2$.

Easy! Suppose the contrary. But then $\chi(M) = \frac{1}{2}\chi(\partial M) = \frac{1}{2}\chi(\mathbb{R}P^2) = \frac{1}{2}$. Contradiction! ■

Fall 2015, 1 and Fall 2020, 10 Let $M_n(\mathbb{R})$ be the space of $n \times n$ real matrices.
(a) Show that $SL_n(\mathbb{R})$ is a smooth submanifold of $M_n(\mathbb{R})$.
(b) Show that $SL_n(\mathbb{R})$ has trivial Euler characteristic.

(a) Consider $\det : M_n(\mathbb{R}) \to \mathbb{R}$. To show that $SL_n(\mathbb{R})$ is a manifold, it's enough to show that 1 is a regular value. To do this, we wish to show that $d(\det)_A$ is nonzero (thus surjective) for $\det(A) = 1$.

We have

$$\begin{aligned} d(\det)_A &= \lim_{h\to 0} \frac{\det(A+hB) - \det(A)}{h} \\ &= \lim_{h\to 0} \frac{\det(A)(\det(I + hBA^{-1}) - 1)}{h} \end{aligned}$$

Now choosing $B = A$, since $\det(A) = 1$, we can see that $d(\det)_A = 1 \neq 0$ as desired. Note that if the derivative is defined, we should be able to take the limit from any direction and obtain the same value.

(b) We show that $SL_n(\mathbb{R})$ is homotopy equivalent to $SO(n)$, and since Euler characteristic is invariant under homotopy, $\chi(SL_n(\mathbb{R})) = \chi(O(n))$. Let $r : M_n(\mathbb{R}) \to SO(n)$ by $A = UP \mapsto U$, where UP is the polar decomposition of A so that P is positive definite and U is unitary and therefore $U \in SO(n)$. Let $i : SO(n) \to M_n(\mathbb{R})$ be the inclusion. By uniqueness of the polar decomposition, $i \circ r = id$, so we show that $r \circ i$ is homotopy equivalent to id. To do this, consider $H_t : SL_n(\mathbb{R}) \to SL_n(\mathbb{R})$ defined by $H_t(A) = \frac{(1-t)A+tU}{\det((1-t)A+tU)}$. We note $\det((1-t)A + tU) \neq 0$ as

$$(1-t)A + tU = U((1-t)P + tI).$$

As P is a positive definite matrix, and the convex combination of positive definite matrices is positive definite, and $\det(U) \neq 0$, we have that $\det((1-t)A + tU) \neq 0$. Additionally, note $t, \det(H_t(A)) = 1$. Moreover, we note that $H_0 = \text{id}$ and $H_1 = r \circ i$, as desired.

Since $SO(n)$ and $SL_n(\mathbb{R})$ are homotopy equivalent, they have the same Euler characteristic. Moreover, as $SO(n)$ is a lie group, it is parallelizable and thus admits a nowhere vanishing vector field. Since $SO(n)$ is closed (it's the inverse image of $\{1\}$ for the map $A \mapsto AA^T$) and bounded, Poincare-Hopf implies $\chi(SO(n)) = 0$.

■

Spring 2017, 3 Use the Poincare-Hopf index theorem to calculate the Euler characteristic of S^n. (You must compute the indices in local coordinates. Drawings do not suffice!)

First, consider $n = 2k-1$, where we then consider the vector field at $p = (a_1, b_1, \cdots, a_k, b_k)$ to be $(-b_1, a_1, \cdots, -b_k, a_k)$. Since $(-b_1, a_1, \cdots, -b_k, a_k)$ is orthogonal to $(a_1, b_1, \cdots, a_k, b_k)$, and thus $(-b_1, a_1, \cdots, -b_k, a_k) \in T_p(S^n)$. Since this vector field has no zeros on S^n, by the Poincare Hopf theorem, we see that S^n for n odd has Euler characteristic 0.

Now, consider $n = 2k$, in which case, S^n sits in $\mathbb{R}^{2k+1}$. For $p = (a_1, b_1, \cdots a_k, b_k, r)$, we define the vector field X at this point to be $(-b_1, a_1, \cdots, -b_k, a_k, 0)$. As before, we note that the output is orthogonal to the input, and thus $(-b_1, a_1, \cdots, -b_k, a_k, 0) \in T_p(S^n)$. We note that in this case, there are two zeros: namely at the points $(0, \cdots, 0, \pm 1)$.

The index of X around $p = (0, \cdots, 0, 1)$ can be computed by taking the degree of $X_p / \|X_p\|$ along a circle around p not containing any other zeroes of X. In particular, we can take the one at the equator (ie. points of the form $(z_1, \cdots, z_k, 0)$ - we note that in this case, $\|X_p\| = 1$, so we are just doing degree of X_p really). We notice that this is a map from S^{n-1} to S^{n-1} taking $(a_1, b_1, \cdots, a_k, b_k) \mapsto (-b_1, a_1, \cdots, -b_k, a_k)$.

Since each sign change contributes a factor of -1 , as do the swaps. Thus, the index here is $(-1)^{n/2}(-1)^{n/2} = (-1)^n = 1$.

The index around $(0, \cdots, 0, -1)$ can be computed the same way, thus giving us a sum of indices of 2. It thus follows then that the Euler characteristic is 2 . ∎

Fall 2017, 7 Suppose M is smooth, connected, and oriented manifold without boundary.
(a) Show that if $\chi(M) = 0$, then M admits a nowhere vanishing vector field.
(b) If M is a surface of genus g, then what is the $\min_v$(number of zeros of v), where v ranges over vector fields whose zeros are isolated and have index ± 1?

(a) Let X be a vector field with a finite number of isolated zeros. We can assume, without loss of generality, that these are located in an open set diffeomorphic to $B_1(0)$ (the unit ball in $\mathbb{R}^n$). Now, we note that since the Euler characteristic is 0, the sum of indices is also zero (by Poincare Hopf), which means that $\deg\left(X_p/\left\|X_p\right\|\right)$ is also zero (viewing this as a function on the boundary of $\bar{U}$ to S^{n-1}), which by the Extension theorem (Guillemin and Pollack page 146) and the Poincare Hopf Index Theorem allows us to extend $X_p/\left\|X_p\right\|$ to g defined on all of $\bar{U}$. Note that f From here, define a vector field Y such that

$$Y_p = \begin{cases} g(p) & p \in \bar{U} \\ X_p/\left\|X_p\right\| & p \notin \bar{U} \end{cases}$$

Thus, Y is a nonvanishing vector field, as desired.

(b) We first notice that M has Euler characteristic $2-2g$, which by Poincare Hopf, given X that has isolated zeros of index ± 1, the sum of indices give $2-2g$. In order to minimize the number of zeros, we must have $2g-2$ zeros all with index -1. This can be done as follows: we know that there is a vector field X with a source, a sink, and $2g$ saddles (if we consider the flow of 'dunking a n-hole donut and lifting it'). Now, we can, without loss of generality, assume that the source, sink, and two of the saddles are contained in a neighborhood U diffeomorphic to an open ball in $\mathbb{R}^2$. Now, we can consider $X_p/\left\|X_p\right\|$ as before, which is nonzero on ∂U (which is diffeomorphic to S^1); since the sum of indices of the source, sink, and two saddles add up to 0, we see that $X_p/\left\|X_p\right\|$ is a degree 0 map on ∂U, which means we can extend it to a map g on all of U.

Unfortunately, we cannot do as we did before, since X_p has zeros in $M \backslash \bar{U}$. To remedy this, we first, assume without loss of generality, that U is the restriction of an open ball $\tilde{U}$ in $\mathbb{R}^3$ (note M can be embedded into $\mathbb{R}^3$) of radius r, and find $\epsilon > 0$ such that the (closed) ball W of radius $r+\epsilon$ centered around the the same point as $\tilde{U}$, contain only the source, sink, and same two saddles. Now, we can use a bump function to define our vector field as follows: outside of $W, Y_p = X_p$, inside of $U, Y_p = g(p)$, and in between, we have a smooth transition that is nonzero. Thus, we have ourselves vector field with only $2g-2$ zeros (all of index 1), since we removed the source, sink, and two saddles.

∎

5 Poincare Duality

Poincare Duality has generally been tested in two ways on previous quals; either it has been used simplify computation involving the rank of cohomology groups or the Euler characteristic, or you have been asked to find Poincare duals to certain submanifolds. In both types of problems, only the relevant results and needed; the proofs likely go beyond the scope of the exam. Therefore, I will present the relevant results here and do my best to highlight how I think you should understand the results in context, but I will not go into much rigorous detail on their proofs. These notes are a combination of Lee's Smooth Manifolds and Hatcher's Algebraic Topology.

Also importantly, hypotheses matter here. There are many related statements of Poincare duality, many with slightly relaxed conditions, and it is important to understand when you can apply a result.

First, let's consider the case of de Rham cohomology, i.e., cohomology over $\mathbb{R}$. Let M be an oriented smooth n-manifold without boundary. We define a map $\mathrm{PD} : \Omega^p(M) \to \Omega_c^{n-p}(M)^*$ via

$$\mathrm{PD}(\omega) : \eta \mapsto \int_M \omega \wedge \eta$$

This descends to a linear map $\mathrm{PD} : H^p(M) \to H_c^{n-p}(M)^*$ because the integrals of exact forms will vanish. Poincare duality says that this map is an isomorphism.

Theorem 5.1 (Poincare Duality)**.** *Let M be an oriented smooth n-manifold without boundary. The map $PD : H^p(M) \xrightarrow{\cong} H_c^{n-p}(M)^*$ is an isomorphism. In particular,* $\dim H^p(M) = \dim H_c^{n-p}(M)$.

This implies that when the cohomologies are finite dimensional that $H^p(M) \cong H_c^{n-p}(M)$, although note that there is no natural isomorphism here because the dual space $H_c^{n-p}(M)^*$ is not naturally isomorphic to $H_c^{n-p}(M)$. So if you are asked to work with the isomorphism explicitly, (finding Poincare duals is related to this) you should use this map.

There is also a related corollary for compact manifolds, where $H^k(M) = H_c^k(M)$. Here we have

Corollary 5.1.1 (Compact Poincare Duality)**.** *Let M be a closed oriented smooth n-manifold. The map $PD : H^p(M) \xrightarrow{\cong} H^{n-p}(M)^*$ is an isomorphism. In particular,* $\dim H^p(M) = \dim H^{n-p}(M)$.

This allows us to prove a result we have seen before. Let M be a closed manifold of odd dimension n; then $\chi(M) = 0$. This is because

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \dim H^p(M) = \sum_{p=0}^{n} (-1)^p \dim H^{n-p}(M) = -\sum_{p=0}^{n} (-1)^p \dim H^p(M)$$

by re-indexing. So $\chi(M) = -\chi(M)$, hence $\chi(M) = 0$. This type of argument is very standard.

Now recall that $H^{n-p}(M)^* = \mathrm{Hom}(H^{n-p}(M), \mathbb{R})$. From the Universal Coefficient Theorem, we see that this is exactly $H_{n-p}(M)$, since $\mathbb{R}$ is free. This gives us an equivalent formulation.

Theorem 5.2 (Poincare Duality Again)**.** *Let M be an oriented n-manifold without boundary. There is an isomorphism $H_c^p(M;\mathbb{R}) \xrightarrow{\cong} H_{n-p}(M;\mathbb{R})$ is an isomorphism. In fact, if M is R-oriented, then there is an isomorphism $H_c^p(M;R) \xrightarrow{\cong} H_{n-p}(M;R)$. In particular,* $\dim H_c^p(M;R) = \dim H_{n-p}(M;R)$.

There is also a simpler version when the manifold is compact. This proof naturally comes from algebraic topology, (hence we no longer need the smooth hypothesis. (Also since cohomology is homotopy invariant)) since we are no longer using the de Rham cohomology groups. The explicit map is now defined using the cap product and the fundamental class. Specifically, if $[\omega] \in H_n(M; R)$ is the fundamental class, then the isomorphism $D : H^p(M; R) \to H_{n-p}(M; R)$ is given by $D : \eta \mapsto [\omega] \cap \eta$.

There is also a generalization of Poincare duality to manifolds with boundary known as Lefschetz Duality.

Theorem 5.3 (Lefschetz Duality)**.** *Let M be a compact R-orientable n-manifold with boundary ∂M. Then there is an isomorphism $H^p(M, \partial M; R) \xrightarrow{\cong} H_{n-p}(M; R)$.*

More generally, if we can decompose ∂M into the union of two compact $(n-1)$-manifolds A and B with common boundary $\partial A = \partial B = A \cap B$, then the is an isomorphism $H^p(M, A; R) \xrightarrow{\cong} H_{n-p}(M, B; R)$.

The special case comes by taking $A = \partial M$ and $B = \emptyset$, and this is the most any problem on the qual has previously needed.

An example corollary of this result is that if M is an n-dimensional manifold with boundary, then $H_n(M) = 0$. From Lefschetz duality, we see that $H_n(M) = H^0(M, \partial M) = 0$.

5.1 Universal Coefficient theorem

A result that is often useful for problems like this is the Universal coefficient theorem. This result tells us that $H_i(X; \mathbb{Z})$ and $H^i(X; \mathbb{Z})$ completely determines $H_i(X; A)$ and $H^i(X; A)$ for any other abelian group A.

Theorem 5.4. *(Universal coefficient theorem for homology and cohomology)*

In the homology case, consider the tensor product of modules $H_i(X; \mathbb{Z}) \otimes A)$. Then there is a short exact sequence with the Tor functor

$$0 \to H_i(X; \mathbb{Z}) \otimes A \xrightarrow{\mu} H_i(X; A) \to Tor_1(H_{i-1}(X; \mathbb{Z}), A) \to 0$$

On the other hand, in the cohomology case we assert there is the following short exact sequence with the Ext functor

$$0 \to Ext^1_R(H_{i-1}(X; R), G) \to H^1(X; G) \xrightarrow{h} \hom_R(H_i(X; R), G) \to 0$$

with this we can prove the following corollary of the Poicare Duality.

Corollary 5.4.1 (Corollary 3.37 of Hatcher)**.** *A closed manifold of odd dimension has Euler characteristic zero.*

Proof. Let M be a closed n-manifold. If M is orientable, we have $\operatorname{rank} H_i(M; \mathbb{Z}) = \operatorname{rank} H^{n-i}(M; \mathbb{Z})$ Poincare duality. On the other hand, by exactness in the universal coefficient theorem for cohomology, we know that $H^1(X; G) = Ext^1_R(H_{i-1}(X; R), G) \otimes \hom_R(H_i(X; R), G)$. The Ext contribution to H^{n-i} will be torsion, whereas the Hom contribution to H^{n-i} will be the free part of H_{n-i}. Therefore, $\operatorname{rank} H^{n-i}(M; \mathbb{Z}) = \operatorname{rank} H_{n-i}(M; \mathbb{Z})$. Thus, if n is odd, all the terms of $\sum_i (-1)^i \operatorname{rank} H_i(M; \mathbb{Z})$ cancel in pairs. □

Let's warmup by solving this old friend again!

Fall 2012, #8 Show that there is no compact three-dimensional manifold whose boundary is $\mathbb{R}P^2$.

Suppose M is the manifold whose boundary is $\mathbb{R}P^2$. Let N be the double of M (ie. take two disjoint copies of M and identify them by the boundary). Consider the following Mayer Vietoris Sequence

$$\cdots \to H_n(\partial M) \to H_n(M) \oplus H_n(M) \to H_n(N) \to H_{n-1}(\partial M) \to \cdots$$

Then the assumption on M yields,

$$\cdots \to H_n(\mathbb{R}P^2) \to H_n(M) \oplus H_n(M) \to H_n(N) \to H_{n-1}(\partial M) \to \cdots$$

which gives $\chi\left(\mathbb{R}P^2\right) - 2\chi(M) + \chi(N) = 0$. Note that

$$H_n(\mathbb{R}P^2) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and therefore, $\chi\left(\mathbb{R}P^2\right) = 1$.

As N is a compact oriented 3-manifold, we note that it has by utilizing Poincare duality in the corollary above, that $\chi(N) = 0$. So, $\chi\left(\mathbb{R}P^2\right)$ is even, which contradicts the fact that $\chi\left(\mathbb{R}P^2\right) = 1$. ∎

Fall 2012, #7 and Spring 2015, #10 and Fall 2021, #5 Let $n \geqslant 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n+2$. Show that $\dim H^{2n+1}(M;\mathbb{R})$ is even.

Consider the map $H^{2n+1}(M) \times H^{2n+1}(M) \to \mathbb{R}$ where $(\omega, \eta) \mapsto \int_M \omega \wedge \eta$. We can easily verify this map is bilinear as follows, $(\omega_1 + \omega_2, \eta) = \int_M (\omega_1 + \omega_2) \wedge \eta = \int_M \omega_1 \wedge \eta + \omega_2 \wedge \eta = \int_M \omega_1 \wedge \eta + \int_M \omega_2 \wedge \eta = (\omega_1 + \omega_2, \eta)$ and the η component can be treated similarly. On the other hand, for any $c \in \mathbb{R}$, we know that $(c\omega, \eta) = \int_M (c\omega) \wedge \eta = c \int_M \omega \wedge \eta = c(\omega, \eta)$

This bilinear form is also anti-symmetric since

$$\int_M \omega \wedge \eta = -\int_M \eta \wedge \omega$$

as both $\eta, \omega \in \Omega^{2n+1}(M)$ are both odd-dimensional.

Since $\mathbb{R}$ is a field, Poincare Duality gives us this is non-degenerate (proposition of 3.38 of hatcher).

It follows that $\dim H^{2n+1}(M)$ is even as if it were odd, then

$$\det(A) = \det(A^T) = \det(-A) = (-1)^{\dim H^{2n+1}(M)} \det(A) = -\det(A).$$

However as A is non-degenerate, we know that $\det(A) \neq 0$. Thus, $\dim H^{2n+1}(M)$ is even. ∎

Spring 2019, #10 Suppose M^n is a compact, connected orientable topological manifold with boundary a rational sphere, i.e. with $H_*(\partial M; \mathbb{Q}) \cong H_*(S^{n-1}; \mathbb{Q})$.

(a) Assuming n is odd, use Poincare duality (with $\mathbb{Q}$ coefficients) to show that M has Euler characteristic $\chi(M) = 1$.

(b) Assuming $n \equiv 2 \pmod 4$, show that the Euler characteristic of M is odd.

(a) Similar set up to F2012, #8. Let $N = M \cup_{\partial M} M$. We note that by alternating sum of short exact sequences of the Mayer-Vietoris sequence,

$$\chi(N) - 2\chi(M) + \chi(\partial M) = 0.$$

As N is an odd dimensional closed orientable manifold, Poincare Duality implies $\chi(N) = 0$. We also note that $\chi(\partial M) = \chi(S^{n-1})$, and as n is odd, $\chi(S^{n-1}) = 2$, so $\chi(M) = 1$.

(b) Let $n = 4k+2$. Firstly, we note that $H^n(M) \cong H_0(M, \partial M; \mathbb{Q}) = 0$ and $H^n(M, \partial M \cong H_0(M; \mathbb{Q}) \cong \mathbb{Q}$ by Lefschetz Duality. We analyze the LES of the pair $(M, \partial M)$ in $\mathbb{Q}$ coefficients. As $H_*(\partial M; \mathbb{Q}) \cong H_*(S^{n-1}; \mathbb{Q})$, we have isomorphisms $H^i(M; \mathbb{Q}) \cong H^i(M, \partial M; \mathbb{Q})$ for all $i \leqslant n-1$. Thus, for all $i \leqslant n-1$, we have

$$H^i(M; \mathbb{Q}) \cong H^i(M, \partial M; \mathbb{Q}) \cong H^{n-i}(M; \mathbb{Q}),$$

via Lefschetz Duality. Thus, we obtain

$$\begin{aligned}
\chi(M) &= \sum_{i=0}^{4k+2} (-1)^i \operatorname{rank}(H^i(M; \mathbb{Q})) \\
&= \operatorname{rank}(H^0(M; \mathbb{Q})) + \sum_{i=1}^{2k} (-1)^i \operatorname{rank}(H^i(M; \mathbb{Q})) - \operatorname{rank}(H^{2k+1}(M; \mathbb{Q})) \\
&\qquad\qquad + \sum_{i=2k+2}^{4k+1} (-1)^i (-1)^i \operatorname{rank}(H^i(M; \mathbb{Q})) + \operatorname{rank}(H^{4k+2}(M; \mathbb{Q})) \\
&= 1 + \sum_{i=1}^{2k} (-1)^i \operatorname{rank}(H^i(M; \mathbb{Q})) - \operatorname{rank}(H^{2k+1}(M; \mathbb{Q})) + \sum_{i=1}^{2k} (-1)^i \operatorname{rank}(H^{n-i}(M; \mathbb{Q})) \\
&= 1 + 2\sum_{i=1}^{2k} (-1)^i \operatorname{rank}(H^i(M; \mathbb{Q})) - \operatorname{rank}(H^{2k+1}(M; \mathbb{Q})).
\end{aligned}$$

It suffices to show $\operatorname{rank}(H^{2k+1}(M; \mathbb{Q}))$ is even. However, again, Lefschetz Duality gives us a non-degenerate pairing

$$H^{2k+1}(M; \mathbb{Q}) \otimes H^{2k+1}(M, \partial M; \mathbb{Q}) \to H^{4k+2}(M, \partial M; \mathbb{Q}) \cong \mathbb{Q},$$

given by the cup product. As $H^{2k+1}(M, \partial M; \mathbb{Q}) \cong H^{2k+1}(M; \mathbb{Q})$, composing this isomorphism gives a non-degenerate pairing

$$H^{2k+1}(M; \mathbb{Q}) \otimes H^{2k+1}(M; \mathbb{Q}) \to H^{4k+2}(M, \partial M; \mathbb{Q}) \cong \mathbb{Q}.$$

As $2k+1$ is odd, this is a skew-symmetric non-degenerate bilinear form, implying that $H^{2k+1}(M; \mathbb{Q})$ must be an even dimensional space, as desired.

$\blacksquare$

5.2 Poincare Duals

Despite the linguistic similarity, a Poincare dual is slightly different from the statement of Poincare duality. Put directly, a Poincare dual to a submanifold is a cohomology class (of the codimension of the manifold) that allows integration on the submanifold to be related to integration on the manifold. These notes come entirely from Petersen's Manifold Theory.

Let $S^k \subset N^n$ be a closed oriented submanifold of an oriented manifold with finite dimensional de Rham cohomology of codimension $m = n - k$. A Poincare dual to S is a cohomology class $[\eta_S^N] \in H_c^m(N)$ such that

$$\int_S \omega = \int_N \eta_S^N \wedge \omega$$

for all $\omega \in H^k(N)$. (We will sometimes call a Poincare dual any representative $\eta_S^N \in [\eta_S^N]$.

The cluttered notation suggests that the dependence of the Poincare dual on the ambient manifold N is annoying, and this is true because a Poincare dual might not even exist! For example, N might have no cohomology in dimension m, like $N = S^n$. However, it is true that we can find some neighborhood U of S, because $\int_S \omega$ only depends on the values of ω in a neighorhood of S.

In Petersen's notes, he selects a tubular neighborhood U of S with a deformation retraction $\pi : U \to S$, where the fibers $\pi^{-1}(p)$ are diffeomorphic to $\mathbb{R}^m$ for all $p \in S$. This means that there is an isomorphism

$$\pi^* : H^k(S) \to H^k(U)$$

and we can always find a Poincare dual $[\eta_S^U] \in H_c^m(U)$. That is,

$$\int_S \omega = \int_U \eta_S^U \wedge \pi^*(\omega)$$

This is essentially the same for all tubular neighborhoods, so unless otherwise specified we say that $[\eta_S]$ is a Poincare dual to S in some tubular neighborhood of S. But when this has been asked on previous quals, it has always been the case that a Poincare dual can be found on the entire manifold N.

Spring 2014, #5 Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which respresents the Poincare dual of S.

Let ω denote a representative of the class of the Poincaré dual of S. Since $H^1_{\text{dR}}(M) \cong \mathbb{R}\, dx \oplus \mathbb{R}\, dy$, we have $\omega = a\, dx + b\, dy$ for real numbers a, b. We calculate a, b using the definition of S and of the Poincaré dual.

Let ι denote the inclusion map $S \hookrightarrow M$. Then

$$3 = \int_S \iota^*(dy) = \int_M \omega \wedge dy = \int_M -a\, dx \wedge dy = a$$

and

$$7 = \int_S \iota^*(dx) = \int_M \omega \wedge dx = \int_M b\, dx \wedge dy = -b,$$

where we compute the integrals over S by pulling back to the line segment $\ell \subset L$ from $(0,0)$ to $(7,3)$ in $\mathbb{R}^2$. We conclude that $\omega = 3\, dx - 7\, dy$. ∎

Fall 2015, #4 Let $M = \mathbb{R}^3/\mathbb{Z}^3$ be a three-dimensional torus and $C = \pi(L)$, where $L \subset \mathbb{R}^3$ is the oriented line segment from $(0,1,1)$ to $(1,3,5)$ and $\pi : \mathbb{R}^3 \to M$ is the quotient map. Find a differential form on M which represents the Poincare dual of C.

This problem is very similar to the previous one, but we've moved up a dimension. Let ω denote a 2-form on M representing the Poincaré dual to C. Since (by e.g. Künneth for $S^1 \times S^1 \times S^1$) we have $H^2_{\mathrm{dR}}(M) \cong \mathbb{R}\, dx \wedge dy \oplus \mathbb{R}\, dy \wedge dz \oplus \mathbb{R}\, dx \wedge dz$, we can write $\omega = a\, dx \wedge dy + b\, dy \wedge dz + c\, dx \wedge dz$ for real numbers a, b, c. Now, for $\iota : C \hookrightarrow M$ the inclusion, compute

$$1 = \int_C \iota^*(dx) = \int_M b\, dy \wedge dz \wedge dx = \int_M b\, dx \wedge dy \wedge dz = b,$$

as well as

$$2 = \int_C \iota^*(dy) = \int_M c\, dx \wedge dz \wedge dy = -\int_M c\, dx \wedge dy \wedge dz = -c$$

and

$$4 = \int_C \iota^*(dz) = \int_M a\, dx \wedge dy \wedge dz = a.$$

Again we compute the three leftmost inequalities by pulling back along the parametrization of C. We conclude that $\omega = 4\, dx \wedge dy + dy \wedge dz - 2\, dx \wedge dz$. ■

Fall 2016, #5

(a) Let M be a smooth compact manifold and $N \subset M$ a smooth compact submanifold. Explain (in terms of integrals, without reference to cohomology) what it means for a closed differential form ω to be Poincare dual to N.

In parts (b) and (c), you are free to use your knowledge of homology and cohomology.

(b) Let $M = T^2$ be the two dimensional torus with coordinates $(x, y) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong T^2$. Identify a submanifold $N \subset M$ Poincare dual to the form dy, and show that they are indeed dual.

(c) Give an example of a closed 1-form on T^2 that is not Poincare dual to any submanifold.

We need M and N to be oriented as well. The solution to part (c) is taken from Harris Khan's solutions document.

(a) Let $\iota : N \hookrightarrow M$ be the inclusion, and set $m = \dim M$, $n = \dim N$, and $k = m - n$. A closed form $\omega \in \Omega^k(M)$ is *Poincaré dual* to N if for all closed n-forms η on M we have

$$\int_N \iota^*(\eta) = \int_M \omega \wedge \eta.$$

(b) Let $L \subset \mathbb{R}^2$ be the oriented line segment from $(1, 0)$ to $(0, 0)$, and let S be the image of L under the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 \cong T^2$. We claim dy is Poincaré dual to S. Indeed, for a closed 1-form $a\, dx + b\, dy$ on M representing a general class in $H^1_{\mathrm{dR}}(M)$, and $\iota : S \hookrightarrow M$ the inclusion, we have

$$-a = \int_S \iota^*(\omega),$$

while

$$\int_M dy \wedge \omega = \int_M dy \wedge (a\, dx) = -\int_M a\, dx \wedge dy = -a.$$

Since these are equal, the form dy is Poincaré dual to S as desired.

(c) We claim the closed 1-form $\pi\, dx$ is not Poincaré dual to any closed, oriented submanifold of M. Let $\iota : N \hookrightarrow M$ be the inclusion of a closed, oriented 1-submanifold of M into M, and let $\pi_2 : M \to S^1$ denote projection onto y-coordinate (identifying $M \cong \mathbb{R}^2/\mathbb{Z}^2 \cong (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z}) \cong S^1 \times S^1$). Then if $\pi\, dx$ were Poincaré dual to N, we'd have

$$\int_N \iota^*(dy) = \int_M \pi\, dx \wedge dy = \pi.$$

But dy on M can be identified with $\pi_2^*(d\theta)$ on S^1, so we also have

$$\int_N \iota^*(dy) = \int_N (\pi_2 \circ \iota)^*(d\theta) = \deg(\pi_2 \circ \iota) \int_{S^1} d\theta = \deg(\pi_2 \circ \iota),$$

where we have used the definition of the degree of a map $N \to S^1$ in terms of de Rham cohomology. From the two display lines we conclude that $\deg(\pi_2 \circ \iota) = \pi$, which is impossible since the degree of a map is always an integer. We conclude that $\pi\, dx$ is not dual to any closed oriented submanifold of M.

■

6 Symplectic Forms

Sam's notes give a great concise summary here that I think should be the primary reference here, but I'm going to write down the main results here for completeness.

A symplectic form on a $2n$-dimensional manifold is a closed 2-form $\omega \in \Omega^2(M)$ such that ω^n is nowhere zero, i.e., ω^n is a volume form. The pair (M, ω) is called a symplectic manifold.

The two prototypical examples are $M = \mathbb{R}^{2n}$, where the symplectic form is $\omega = \sum dx_i \wedge dy_i$, and S^2 where any volume form is a symplectic form.

Symplectic manifolds are orientable, because the top cohomology is nontrivial. This means that we can use Poincare duality, with suitable other hypothesis. For example, if (M, ω) is a closed symplectic $2n$-manifold, then $H^{2k}(M) \neq 0$ for all $k \leqslant n$. This is just because $[\omega^n] \neq 0$, and $[\omega^n] = [\omega^k] \wedge [\omega^{n-k}]$.

Spring 2018, #5 A symplectic form on an eight dimensional manifold is a closed 2-form ω such that ω^4 is a volume form. Determine which of the following admits a symplectic form; $S^8, S^2 \times S^6, S^2 \times S^2 \times S^2 \times S^2$.

We analyze each given manifold separately. Note that the de Rham cohomology of S^n satisfies

$$H^p_{dR}(\mathrm{S}^n) = \begin{cases} \mathbb{R} \text{ if } p = 0, \text{ of } p = n \\ 0 \text{ if } 0 < p < n \end{cases}$$

and that the ring cohomology is

$$H^*(S^n) \equiv \mathbb{Z}[\alpha]/(\alpha^2), \quad |\alpha| = n$$

- S^8: Since $H^2_{dR}\left(S^8\right) = 0$, any closed 2-form is exact, meaning that on the level of homology, for any closed 2-form $\omega, \omega^4 = 0$ cannot be a volume form.

- $S^2 \times S^6$: Consider $H_{dR}^*\left(S^2 \times S^6\right) \cong H_{dR}^*\left(S^2\right) \otimes H_{dR}^*\left(S^6\right) \cong \mathbb{C}\left[s^2, s^6\right] / \left(s^4, s^{12}\right)$. We note that the elements of grading 2 all have the form cs^2 for some c, which means for a closed 2-form ω, we have $\omega \wedge \omega = 0$, and so we cannot possibly have ω^4 being a volume form here either.
- $S^2 \times S^2 \times S^2 \times S^2$: Finally, note that $H_{dR}^*\left(S^2 \times S^2 \times S^2 \times S^2\right) \cong \mathbb{C}\left[s^2, t^2, u^2, v^2\right] / \left(s^4, t^4, u^4, v^4\right)$. Consider $\left(s^2 + t^2 + u^2 + v^2\right)^4$, which is of the form $cs^2t^2u^2v^2$, for some nonzero c. We note that this is a generator of H^8, meaning that this is a volume form.

∎

Spring 2020, #2 Let M be a 4-dimensional manifold. A symplectic form is a closed 2-form ω such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.
(a) Construct a symplectic form on $\mathbb{R}^4$.
(b) Show that there are no symplectic forms on the unit sphere S^4.

(a) Since we are in $\mathbb{R}^4$, we want to construct a form such that ω^2 is nowhere zero. Consider $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ we get $\omega \wedge \omega = 2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$

(b) We note that any closed 2 form ω is exact as $H^2(S^4) = 0$. It thus follows that $\omega \wedge \omega$ is also exact for any 2 form ω since if $\omega = dv$ then $d(v \wedge dv) = dv \wedge dv = \omega \wedge \omega$. thus we can see that $\omega \wedge \omega$, then by continuity, $\int_{S^4} \omega \wedge \omega \neq 0$, giving us our desired contradiction.

∎

Spring 2022, #1 Let M be a closed (compact, without boundary) $2n$-dimensional manifold, and let ω be a closed 2-form on M which is non-degenerate, i.e., for any $p \in M$, the map $T_pM \to T_p^*M$, $X \to i_X\omega(p)$ is an isomorphism. Show that the de Rham cohomology groups $H_{dR}^{2k} \neq 0$ for $0 \leqslant k \leqslant n$.

It suffices to show $H_{dR}^{2n} \neq 0$, as $[\omega^n] = [\omega^k] \wedge [\omega^{n-k}]$ for any $0 \leqslant k \leqslant n$. Consider the map $T_pM \times T_pM \to \mathbb{R}$ where

$$X \times Y \mapsto \omega(p)(X, Y).$$

This is a bilinear form as $\omega(p)$ is a multilinear map. However as it is alternating, we have

$$\omega(p)(X, Y) = -\omega(p)(Y, X).$$

Finally, we see ω is non-degenerate because $Y \to \omega(p)(X, Y)$ is exactly the map $i_X\omega(p)$, which is an isomorphism by hypothesis. That is, for any $X \in T_pM$, $\exists Y \in T_pM$ such that $\omega(p)(X, Y)$ is nonzero, which proves $\omega(p)$ is non-degenerate.

So we have a non-degenerate bilinear skew-symmetric form. Thus, there exists a basis

$$X_1, \ldots, X_n, Y_1, \ldots, Y_n \in T_pM$$

such that

$$\omega(p)(X_i, Y_j) = \delta_{ij},$$

and

$$\omega(p)(X_i, X_j) = \omega(p)(Y_i, Y_j) = 0.$$

Thus,

$$\omega(p) = \sum_{i=1}^{n} X_i^* \wedge Y_i^*,$$

implying

$$\omega^n(p) = n!\, X_1^* \wedge \cdots \wedge X_n^* \wedge Y_1^* \wedge \cdots \wedge Y_n^*.$$

Thus,

$$\omega^n(p)(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = n!$$

implying ω^n is nowhere vanishing as $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ is a basis for T_pM, implying M is orientable. As ω^n is nowhere vanishing, $\int_M \omega^n \neq 0$, and as M is closed, this implies ω^n is not exact, as desired. ∎

Fall 2022, #5 Let M be a $2n$-dimensional manifold. A symplectic form on M is a smooth closed 2-form in $\Omega^2(M)$ so that $\omega \wedge \ldots \wedge \omega \in \Omega^{2n}(M)$ is a volume form. (That is, nowhere vanishing) Determine all pairs of positive integers (k, ℓ) so that $S^k \times S^\ell$ has a symplectic form.

We show the only pairs are $k = \ell = 1$ and $k = \ell = 2$.

Note that if $S^k \times S^\ell$ is a symplectic form, as $S^k \times S^\ell$ is closed, we require ω^n to not be exact. As $[\omega^n] = [\omega^k] \wedge [\omega^{n-k}]$, this implies we require all even De Rham cohomologies to be nontrivial. We also need $k + \ell$ to be even.

Suppose $k > 2$ and $\ell > 2$. Then, by Künneth's formula,

$$\begin{aligned} H^2(S^k \times S^\ell) &= H^2(S^k) \otimes H^0(S^\ell) \oplus H^1(S^k) \otimes H^1(S^\ell) \oplus H^2(S^k) \otimes H^0(S^\ell) \\ &= 0 \otimes \mathbb{Z} \oplus 0 \otimes 0 \oplus 0 \otimes \mathbb{Z} = 0 \end{aligned}$$

So, none of the are possibilities as symplectic manifolds.

Suppose $k = 2$ and $\ell > 4$. Then, by Künneth's formula,

$$\begin{aligned} H^4(S^2 \times S^\ell) &= H^2(S^2) \otimes H^2(S^\ell) \oplus H^1(S^2) \otimes H^3(S^\ell) \oplus H^0(S^2) \otimes H^4(S^\ell) \\ &= \mathbb{Z} \otimes 0 \oplus 0 \otimes 0 \oplus \mathbb{Z} \otimes 0 = 0 \end{aligned}$$

So, none of the are possibilities as symplectic manifolds.

So, the possible candidates are $S^2 \times S^2$, $S^1 \times S^1$, and $S^2 \times S^4$.

$S^1 \times S^1$ is symplectic as it is an orientable two dimensional manifold, as the product of orientable manifolds is orientable, so any volume form is our symplectic form.

We show $S^2 \times S^2$ is symplectic. Let η be a volume form on S^2, which exists as S^2 is orientable, and let $\pi_i : S^2 \times S^2 \to S^2$ be projection onto the ith coordinate. We have, via Künneth, $\pi_1^*\eta \wedge \pi_2^*\eta$ is a volume form on $S^2 \times S^2$. Take $\omega = \pi_1^*\eta + \pi_2^*\eta$. Note that $\pi_i^*\eta \wedge \pi_i^*\eta = \pi_i^*(\eta \wedge \eta) = 0$ as $\eta \wedge \eta$ is a 4-form on S^2. Thus,

$$\omega \wedge \omega = 2\pi_1^*\eta \wedge \pi_2^*\eta,$$

which is a volume form, as desired.

$S^2 \times S^4$ is not symplectic. Suppose it were. Note that, via Künneth,

$$H^2(S^2 \times S^4) = H^2(S^2) \otimes H^0(S^4) \cong \mathbb{Z}.$$

Thus, it is spanned by $\pi_1^*\eta$ where η is a volume form on S^2. Suppose ω were a symplectic form on $S^2 \times S^4$. Then, $[\omega] = c[\pi_1^*\eta]$. Thus, $[\omega^3] = c^3[\pi_1^*\eta^3] = 0$. However this contradicts the fact the ω^3 is a volume form, as it cannot be exact. ∎

Extra Which torii $(S^1)^n$ are symplectic manifolds? Which products $(S^2)^n$ are symplectic manifolds?

7 The Fundamental Group

Hopefully, we already know the basics of the fundamental group but I'll put down the basic definitions and then do more in adding the important propositions/facts surrounding the topic. I'll try to include all the technical definitions where they show up. My source is Hatcher, but I do things slightly out of order, in a way that makes sense to me, who already knows the material and not learning it for the first time.

If we are given a connected topological space X and a specific point $x_0 \in X$, we can define the fundamental group of X based at x_0, $\pi_1(X, x_0) = L(X, x_0)/\sim$ where $L(X, x_0)$ is the space of all the loops based at x_0. The relation $\sim$ is based homotopy equivalence, that is a homotopy $H_t(x)$ where $H_t(1) = x_0 = H_t(0)$ for all $t \in [0, 1]$. We call this a group since it has a group structure where the product of two loops is the concatenation of two loops and the inverse of a loop is running the loop in reverse, and the identity is the equivalence class of the constant loop $\gamma(x) = x_0$.

7.1 Relations of Spaces and Fundamental Groups

I feel like one of the most important things to keep straight is where maps between spaces and fundamental groups arise. Like, if a pair of spaces have some property what does that say about their fundamental groups? Or if I have some property about fundamental groups what does that say about the spaces and the maps between them. In my mind this includes all the covering space stuff.

One of the first questions we could ask is what do maps between two topological spaces $f : X \to Y$ do to the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$ relate? Since any continuous map sends loops to loops, preserves concatenation, homotopy, and sends the trivial loop to the trivial loop, we get that every continuous map f induces a homeomorphism on fundamental groups, also known as a push forward, $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ where $f_*[\gamma] = [f \circ \gamma]$. We can then ask what sort of group homomorphisms between fundamental groups can naturally arise. For example, by considering the projection maps π_X, π_Y from $X \times Y$ to X and Y we get the following:

Proposition 7.1. *If X and Y are path connected spaces, then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

This isomorphism arises naturally by sending $\gamma : I \to X \times Y$ to $(\pi_X(\gamma), \pi_Y(\gamma))$ (here π is the projection map) we can see that the order of the first component vs second component does not matter since that can change by homotopy in $X \times Y$.

One of the most important group homomorphisms, the change of base-point isomorphism, does not arise as a push forward of a map. If γ is a path from x_0 to y_0, then we can consider the map $\beta_h : \pi_1(X, x_0) \to \pi_1(X, y_0)$ which sends a loop γ to $h \cdot \gamma \cdot h^{-1}$ where $\cdot$ represents concatenation of paths, we can check that this obeys all the group laws and has an inverse which is $\beta_{h^{-1}}$. These isomorphisms sometimes show up in propositions (like the next few), so it is good to be comfortable with them.

Since the fundamental group is a topological object it is not too surprising that it is invariant under homotopy, some of the time we want to know how it is invariant, as in exactly what are the isomorphisms which is the topic of the next few propositions.

Proposition 7.2. *If $H_t(x) : I \times X \to Y$ is a homotopy, then we know then the push forwards H_{0*} and H_{1*} are related by the change of base-point isomorphism on Y via the path $h : t \to H_t(0)$. That is $H_{1*} = \beta_h \circ H_{0*}$*

Proposition 7.3. *If $f : X \to Y$ is a homotopy equivalence, then $f_* \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is a group isomorphism.*

This second follows from the first since if $f \circ g \simeq \mathbb{1}$ then which implies that $f_* \circ g_* = (f \circ g)_* = \mathbb{1}_* \circ \beta_{h*} = \beta_{h*}$ so we can see that $f_* \circ g_*$ and $g_* \circ f_*$ are group isomorphisms of $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$ respectively so they are both isomorphisms.

So homotopy equivalent spaces have the same fundamental group. See here for some[1] *clever* examples. We can also ask what happens with the push-forward from covering maps.

Theorem 7.1. *If $p : \tilde{X} \to X$ is a covering map. Then $p_* : \pi_1(\tilde{X}, x_0) \to (X, p(x_0))$ is an injective isomorphism. That is $p_*(\pi_1(\tilde{X}, x_0))$ is a subgroup of $\pi_1(X, p(x_0))$. If X and $\tilde{X}$ are path connected then the number of sheets in this covering space is the index of the subgroup,*

The theorem comes from properties about the homotopy lifting property of covering spaces. This gives us a necessary and sufficient condition for a map $f : X \to Y$ to lift to the covering space $X \to \tilde{Y}$. To do this we will need one more technical definition.

Definition 7.1. *A space X is locally path connected, if for all $x \in X$ and neighborhoods V of x, there is a neighborhood U of x, where $U \subset V$ and U is path connected.*

Proposition 7.4. *Suppose that Y is a path connected space and locally path connected. Then $f : X \to Y$ lifts to $\tilde{f} : X \to \tilde{Y}$ if and only if $f_* \pi_1(X, x_0) < p_*(\tilde{Y}, y_0)$ where $p(y_0) = f(x_0)$.*

The locally path connected definition is used to show that the lift is continuous. That is points near x lift to points near $\tilde{f}(X)$ since a short path in X connects them, so a short path in Y connects them and thus a short path in $\tilde{Y}$ connects them.

Remark 7.1. *We should not confuse this with the other map lifting properties which deal with lifting the domain not the image, that $f : X \to Y$ lifts to $\tilde{f} : \tilde{X} \to Y$ which always exists and is continuous. The more interesting question is when $\tilde{f} : \tilde{X} \to Y$ descends, which is when it is constant on the fibers of p.*

Recall, the uniqueness of lifts, that if $\tilde{f}_1$ and $\tilde{f}_2$ agree at a point then they agree on that entire path component. One special case of lifting maps properties is in the classification of covering spaces.

Proposition 7.5. *If X is path connected and locally path connected, and has covering maps $p_1 : \tilde{X}_1 \to X$ and $p_2 : \tilde{X}_2 \to X$, where $p_{1*}(\tilde{X}_1, x_1) = p_{2*}(\tilde{X}_2, x_2)$ where $p_1(x_1) = p_2(x_2)$, then $\tilde{p_1} : \tilde{X}_1 \to \tilde{X}_2$ which sends x_1 to x_2 is a homeomorphism.*

There is also a converse that if $\tilde{X}_1$ and $\tilde{X}_2$ are covering spaces and $f : \tilde{X}_1 \to \tilde{X}_2$ is an homeomorphism with $f(x_1) = x_2$, then $p_{1*}(\tilde{X}_1, x_1) = p_{2*}(\tilde{X}_2, x_2)$.

[1]URL: https://www.math3ma.com/blog/clever-homotopy-equivalences

We can note that if X is path connected and locally path connected and x_1 and x_2 are both in the fiber of x_0 then $\tilde{p}_*(\tilde{X}, x_1)$ and $\tilde{p}_*(\tilde{X}, x_2)$ **need not be the same**! But they will be isomorphic as subgroups of $\pi_1(X, x_0)$. In fact, we can find what the isomorphism is. If g is loop in X which lifts to a path from x_1 to x_2, then we can see that $\pi_1(\tilde{X}, x_1)$ and $\pi_1(\tilde{X}, x_2)$ are isomorphic by change of base-point transformation $\beta_{\tilde{g}}$ which sends $\tilde{\gamma}$ to $\tilde{g} \circ \tilde{\gamma}\tilde{g}^{-1}$, which corresponds to the map $g \circ \gamma \circ g^{-1}$ in $\pi_1(X, x_0)$ where note that g and g^{-1} are now elements of $\pi_1(X, x_0)$ and not paths so the change of base-point is in fact conjugation. We then get the following important corollary.

Proposition 7.6. *If X is a path connected and locally path connected space, where $p : (\tilde{X}, x_1) \to (X, x_0)$ is a covering space. $p_*(\pi_1(\tilde{X}, x_1))$ is a normal subgroup of $\pi_1(X, x_0)$ if and only if p is a normal covering space (that is the deck transformations are transitive).*

Since the deck-transformations are the homoemorphism of the covering space $\tilde{X}$, then we can note that they form a group of themselves, denoted by $G(\tilde{X})$ in Hatcher. It is also true that $G(\tilde{X})$ is isomorphic to $N(p_*(\pi_1(\tilde{X}, x_1)))/p_*(\pi_1(\tilde{X}, x_1))$ where $N(G)$ is the normalizer of the group G. This is most useful when p is a normal covering so $N(p_*(\pi_1(\tilde{X}, x_1))) = \pi_1(X, x_0)$ and so $G(\tilde{X}) \cong \pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, x_1))$.

We can further classify all the covering spaces if our space has the added property of being semi-locally simply connected, note that being locally simply connected, simply connected, and locally contractible (e.g. CW complexes and manifolds) are a stronger conditions which often hold in real life.

Definition 7.2. *A space X is semi-locally path connected if for all $x \in X$ there is a neighborhood U of x such that the inclusion map $i : U \to X$ has the trivial push-forward $i_*\pi_1(U, x) \to \pi_1(X, x_0)$. That is every path of U is sent to a null homotopic path in X (it need not be null homotopic in U)*

Theorem 7.2. *If a space X is path connected, locally path connected, and semi-locally simply connected then there is a bijection between the subgroups of $\pi_1(X, x_0)$ and the set of path connected covering spaces of X (up to isomorphisms preserving base-points). Notably this isomorphism is realized by $p : \tilde{X} \to X$ by $p_*\pi_1(\tilde{X}, x_0)$ is $\pi_1(X, p(x_0))$.*

This has two big consequences in my mind. One if a space is semi-locally simply connected, then it has a universal cover (which is useful in some qual problems). Or, we can find the set of subgroups of a group by finding the path connected covering spaces of that space.

7.1.1 Group actions and Covering spaces

There is an aside about when group actions lead to covering spaces, this also occurs in smooth manifolds for that I am using Lee. Suppose instead we are given a group action G on a space X and want to know if $X \to X/G$ is a covering space. If we are in the realm of smooth manifolds, we will first give a few definitions.

Definition 7.3. *A smooth group action by a Lie group G on a smooth manifold M is proper if only if $G_K = \{g \in G : (g \cdot K) \cap K)\}$ is compact for all compact $K \subset M$.*

Note if G is a finite group we can give it the discrete topology and every action is proper. If it is infinite, we need only show that G_K is finite for all K.

Proposition 7.7. *If G is a Lie group whose action on a smooth manifold M is smooth, free, and proper. Then $q : M \to M/G$ is a smooth normal covering map.*

If on the other hand we are working in the continuous case we get the following instead.

Proposition 7.8. *If a group G acts on a space X, such that for all $x \in X$ there is a neighborhood U of x such that $G_U = \emptyset$, then $q : X \to X/G$ is a normal covering space. If we assume further that X/G is path connected the deck transformations are isomorphic to G, and if we even further assume X/G is locally path connected then $G \cong \pi_1(X/G)/q_*(\pi_1(X))$*

7.2 Computations

In my head there are two major tools for actually computing fundamental groups. The first is Van-Kampen the second is the computation for CW complexes. I think usually, we mess around with homotopies and homotopy equivalences to reduce something to a space we can apply one of these two tools or use Van-Kampen to break up a space that can be homotoped nicely.

Theorem 7.3 (Van Kampen)**.** *If X is a union of the interiors of path connected sets $\{A_i\}$ each containing a single point x_0. If each pairwise intersection is path connected. Furthermore, if every triple intersection is path connected. Then we have that $\Phi : *_i \pi_1(A_i) \to \pi_1(X)$ is surjective with kernel generated by $i_{\alpha\beta}(\omega)i_{\beta\alpha}^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$*

There a few corollaries of this theorem.

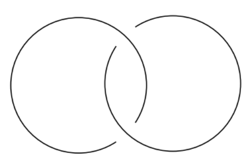
Proposition 7.9. *$\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$ if the wedge point is a deformation retract of a neighborhood in X and Y.*

Proposition 7.10. *If X is a CW complex then we can note that $\pi_1(X)$ is given by $\pi_1(X^1)/N$ where N is generated by the boundaries of the attached 2-cells.*

Proposition 7.11. *$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.*

7.3 Problems

Fall 2013, #9 Let $H \subset S^3$ be the Hopf link, shown in the figure



Compute the fundamental group and the homology groups of the complement $S^3 - H$.

We can use stereographic proejction to see that $S^3 - \{p\}$ is homeomorphic to $\mathbb{R}^3$, while $S^1 - \{p\}$ is homeomorphic to $\mathbb{R}$. Thus, we see that $S^3 - H$ is homeomorphic to $\mathbb{R}^3$ with the z axis and the unit circle $(\cos(2\pi t), \sin(2\pi t), 0)$ removed. We note that this deformation retracts to the torus (this can be seen in that first, this deformation contracts to a sphere with a circle removed, as well as a line that passes through the middle of the circle removed, from which we can enlarge the removed circle as well as the line). From here, we have $S^2 - H$ having the same fundamental group and homology groups as T^2. These are explicitly given as follows

$$\pi_1(S^3 - H) = \pi_1(T^2) = \mathbb{Z} * \mathbb{Z}$$

$$H_n(S^3 - H) = H_n(T^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } n = 1 \\ \mathbb{Z} & \text{if } n = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

■

Fall 2014, #8 Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a Möbius band is the quotient space $([-1,1] \times [-1,1])/((1,y) \sim (-1,-y))$.
(a) Determine the fundamental group of X.
(b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

(a) We proceed by Van Kampen. We'd like to apply van-Kampen to the cover $X = M_1 \cup M_2$, but the issue is that M_1 and M_2 are not open in X. So instead, we take A and B to be small epsilon neighborhoods of M_1 and M_2 and then we apply van-Kampen to the cover $X = A \cup B$.

More explicitly, Let A and B be thickenings of M_1 and M_2 respectively with the identification. In this case, we note that A and B are both homotopic to S^1, while $A \cap B$ is homotopic to S^1 as well, but wrapping around twice. Here, we note that the image of the loop around $A \cap B$ of the map $i_A : A \cap B \hookrightarrow A$ and $i_B : A \cap B \hookrightarrow B$ are homotopic to taking the loop around twice. Thus, we note that $\pi_1(X) \cong \pi_1(A) * \pi_1(B)/N$, where N is generated by $(i_A)_* (\gamma) (i_B)_* (\gamma)^{-1}$. If we let $\pi_1(A) \cong \mathbb{Z}$ be generated by a and $\pi_1(B) \cong \mathbb{Z}$ be generated by B, then N is generated by a^2b^{-2}. So, $\pi_1(X) = \langle a, b \mid a^2b^{-2} \rangle$.

(b) We notice from the above that $H_1(X) = \pi_1(X) = \langle a, b \mid a^2b^{-2}, aba^{-1}b^{-1} \rangle$, the abelianization of $\pi_1(X)$, has torsion elements: namely, ab. On the other hand, we note that $H_1(M_g) = \mathbb{Z}^{2g}$ is torsion free. Thus, for no g do we have X and M_g being homotopy equivalent.

■

Spring 2015, #8 Let X be a CW complex consisting one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a and b map to p, the boundary of f_1 is mapped to the loop ab^2 (that is first a and then b twice), and the boundary of f_2 is mapped to the loop ba^2.
(a) Compute the fundamental group $\pi_1(X)$ of X. Is it a finite group?
(b) Compute the homology groups $H_i(X)$, $i = 0, 1, \ldots,$ of X.

(a) The boundary of two cells f_1 and f_2 contract to a point and thus the fundamental group has the following presentation $\langle a, b \mid ab^2, ba^2 \rangle$. In particular, we note that $a = b^{-2}$, and $b^{-1} = a^2$, meaning that $a = a^4$, so $a^3 = 1$. Similarly, $b^3 = 1$. Moreover, we note that $ab^2 = 1$, so $ab^3 = b$, so $a = b$. Thus, this group is just $\langle a \mid a^3 \rangle$. There is only 1 group of order 3, which is $\mathbb{Z}_3$ and this is certainly finite.

(b) It immediately follows that $H_1(X) = \mathbb{Z}_3$ since the first homology is the abelianization of the fundamental group. We have the following chain complex (starting from 3-simplices),

$$0 \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$$

where the map $\mathbb{Z}^2 \to \mathbb{Z}$ is 0, and since $\partial f_1 = a + 2b$ and $\partial f_2 = 2a + b$ it follows that $\mathbb{Z}^2 \to \mathbb{Z}^2$ is given by $(a, b) \mapsto (a + 2b, 2a + b)$. The kernel of this map is clearly trivial with $a = b = 0$, so $H_2(X) = 0$ as well as all the other higher dimensional homology groups. It is also clear from the above sequence that $H_0(X) = 0$, since it is path connected.

■

8 Covering Spaces

An important result is the following which relates deck transformation (isomorphism between covering spaces) and the fundamental group.

Theorem 8.1. *(Proposition 1.39 of Hatcher) Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path connected covering space of the path connected locally path connected space X, and let H be the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0)$. Then*

(a) This covering space is normal iff H is a normal subgroup of $\pi_1(X, X_0)$

(b) $G(\tilde{X})$ is isomorphic to the quotient $N(H)/H$ where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.

In particular, $G(\tilde{X})$ is isomorphic to $\pi_1(X, x_0)/H$ if $\tilde{X}$ is a normal covering. Hence for the universal cover $\tilde{X} \to X$ we have $G(\tilde{X}) \approx \pi_1(X)$.

8.1 Problems

Fall 2013, #7 Let $M = T^2 - D^2$ be the complement of a disk inside the two-torus. Determine all connected surfaces that can be described as 3-fold covers of M.

Comment on these types of problems: coming up with a cell structure with a single vertex is usually helpful. If this can't be done, contract a maximal subtree to a point (just remember to expand this vertex back out into the full tree at the end...I think this works). Then do the graph classifying business making sure loops given by 2-cell attachments downstairs are still loops in the covering upstairs. Because attaching a 2-cell lets the loop homotope to 0 downstairs and by injectivity of covering map it must do the same upstairs.

Oh and here's the reason the maximal subtree idea needs to work. Homotopy equivalent spaces have the same classes of coverings spaces by the correspondence. So you better not add or remove any coverings by adding your contracted subtree back in, and the only way to do that seems to be to expand it out at the corresponding vertices in the cover.

Fall 2014, #9 Determine all the connected covering spaces of the wedge sum $\mathbb{R}P^{14} \vee \mathbb{R}P^{15}$.

First note that any connected (locally path-connected, semi-locally simply connected) space X admits a simply connected double cover $\tilde{X}$, then its only connected covering spaces are X and $\tilde{X}$ we can see this because $\tilde{X}$ is the universal cover of X, and since it is a double cover, (proposition 1.39 of Hatcher) $\pi_1(X)$ must have order 2 (so must be $\mathbb{Z}/2\mathbb{Z}$) so its only subgroups are the trivial group (corresponding to the universal cover) and the whole group (corresponding to the trivial cover $X \to X$).

Therefore, the covering spaces for $\mathbb{R}P^{14}$ are $\mathbb{R}P^{14}$ and S^{14}, and similarly the covering spaces for $\mathbb{R}P^{15}$ are itself and S^{15}. In particular, in the covering of $X = \mathbb{R}P^{14} \vee \mathbb{R}P^{15}$, when we have a S^{14} or S^{15}, since it's a double cover, there are two connecting points that can be wedge summed with coverings of the other, while $\mathbb{R}P^{14}$ and $\mathbb{R}P^{15}$ has only one of these connecting points. Thus, the covering spaces that we have are as follows: we can have a chain that begins with $\mathbb{R}P^{14}$ or $\mathbb{R}P^{15}$ that ends with the other one (since we need to ensure that we our covering degree is the same), with alternating S^{14} and S^{15} in the middle, we can have an alternating circle of S^{14} and S^{15}, we can have an infinite chain that starts with a $\mathbb{R}P^{14}$ or $\mathbb{R}P^{15}$ that infinitely alternate between S^{14} and S^{15}, and we can also have an infinite chain of alternating S^{14} and S^{15} that have no beginning or end. ∎

Spring 2014, #7 Let X be the wedge sum $S^1 \vee S^1$. Give an example of an irregular covering space $\tilde{X} \to X$.

We recall that the coverings of S^1 are S^1 itself and $\mathbb{R}$. We note that the preimage of the wedge point in the covering space must be wedged with the wedge point of the covering space of the other S^1. Let $\pi_1(X) = \pi_1\left(S^1\right) * \pi_1\left(S^1\right)$, with a corresponding to the first $\pi_1\left(S^1\right)$, and b for the second. Consider $\tilde{X}$ such that we have S^1 (corresponding to a), with wedged with $\mathbb{R}$ (corresponding to b), and each wedge point of $\mathbb{R}$ (corresponding to a) is wedged with $\mathbb{R}$ that corresponds to b, and each subsequent wedge points, are all wedged with $\mathbb{R}$ accordingly. We note that if we choose $\pi_1(\tilde{X}, \tilde{x})$, where $\tilde{x}$ is the wedge point of the initial S^1, then $p_*\pi_1(\tilde{X}, \tilde{x}) = \langle a \rangle$. We note that this is not normal, since given b, we see that $bp_*\pi_1(\tilde{X}, \tilde{x})b^{-1} = \left\langle bab^{-1} \right\rangle$. Thus, we see that this is not a regular covering space. ∎

9 Computing Homology: Chain Complexes and Homology, Mayer-Vietoris, and the Sequence of a Pair

As the obscene number of problems in this section might suggest, being comfortable with the algebraic machinery of chain complexes and homology and with certain long exact sequences in homology can make taking the qual much more pleasant. It's safe to bet money that at least one problem on your qual will fall to these methods, and it's not uncommon to have two or even three exact sequence problems on the same exam. Qual problems that make use of these techniques might ask you to...

- do concrete computations with abstract chain complexes, including demonstrating the existence of certain long exact sequences;
- compute the homology (or more rarely, de Rham cohomology) of some weird quotient space cooked up from nice spaces (by far the most common);
- relate the homology of a suspension to the homology of the space you're suspending (this falls under the previous bullet but comes up so often it's worth mentioning on its own);
- compute relative homology, including homology of certain quotients not falling under the second bullet, using the exact sequence of a pair.

We'll go through these bullets in order, outlining the required definitions and facts as we go.

9.1 Abstract Chain Complexes and Homology

The aim of this section is to very quickly review the formal machinery of chain complexes and homology. There aren't too many problems purely on abstract chain complexes, but knowing how to work with them will net you points should such a problem come up and will also give you more perspective for where the exact sequences we'll see are coming from. In particular, we'll prove the snake lemma and show how to use it to obtain long exact sequences in homology.

Unless otherwise stated, all objects in our chain complexes are abelian groups.

Definition 9.1. *A* chain complex $(A_\bullet, d_\bullet)$ *is a collection of abelian groups* $\ldots, A_0, A_1, \ldots$ *along with maps ("differentials")* $d_i : A_i \to A_{i-1}$ *satisfying* $d_{i-1} \circ d_i = 0$ *for all* i.

A cochain complex *$(A^\bullet, d^\bullet)$ is defined similarly, except the differential d^i now maps $d^i : A^i \to A^{i+1}$ and we require $d^{i+1} \circ d^i = 0$.*

I always remember the indexing on the differentials as matching that of the domain. Maybe "d for domain" is a useful mnemonic for this, maybe not.

Examples include:

- The singular chain complex with $C_i(X)$ defined as the free abelian group on continuous maps $\Delta_i \to X$ for X a space and $i \geqslant 0$. Writing $[v_0, \ldots, v_i]$ for the standard i-simplex, the map $d_i : C_i(X) \to C_{i-1}(X)$ is given by

$$\sigma \mapsto \sum_{j=0}^{i} (-1)^j \sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i]},$$

  where the hat denotes omission.
- The de Rham cochain complex of differential forms on a smooth manifold M. The abelian groups are the groups $\Omega^i(M)$ of differential i-forms on M for $i \geqslant 0$, and the differentials $d^i : \Omega^i(M) \to \Omega^{i+1}(M)$ are given by the exterior derivative.

Definition 9.2. *Let $(A_\bullet, d_\bullet)$ be a chain complex. Its* homology groups *are defined by $H_n = \ker d_n / im\ d_{n+1}$.*

Similarly, for a cochain complex $(A^\bullet, d^\bullet)$, its cohomology groups *are defined by $H^n = \ker d^n / im\ d^{n-1}$.*

In the examples above, we get singular homology $H_i(X)$ and de Rham cohomology $H^i_{\mathrm{dR}}(M)$ as the (co)homology groups of the relevant complexes.

Definition 9.3. *A sequence of abelian groups $A \xrightarrow{f} B \xrightarrow{g} C$ is* exact (at B) *if $\ker g = im\ f$. A longer sequence of abelian groups is* exact *if it is exact at every group in the sequence.*

So, algebraically, homology measures the failure of a chain complex to be an exact sequence at every object of the complex.

To actually compute homology, say the singular homology (resp. de Rham cohomology) of a space (resp. manifold), it's extremely useful to be able to compare homology across chain complexes. In topological contexts, this typically means building a more difficult space or manifold out of simpler ones with known (co)homology in a way that lets us read off the more difficult (co)homology using algebraic means. We get these comparisons using long exact sequences associated with short exact sequences of complexes.

Definition 9.4. *Let $(A_\bullet, d_{\bullet,A})$ and $(B_\bullet, d_{\bullet,B})$ be chain complexes. A* chain map *$f_\bullet : (A_\bullet, d_{\bullet,A}) \to (B_\bullet, d_{\bullet,B})$ is a collection of group homomorphisms $f_i : A_i \to B_i$ for which $d_{i,B} \circ f_i = f_{i-1} \circ d_{i,A}$ for all i. A* short exact sequence *of chain complexes consists of chain complexes $(A_\bullet, d_{\bullet,A})$, $(B_\bullet, d_{\bullet,B})$, and $(C_\bullet, d_{\bullet,C})$ and chain maps $f_\bullet : A_\bullet \to B_\bullet$ and $g_\bullet : B_\bullet \to C_\bullet$ for which the sequences*

$$0 \to A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \to 0$$

are exact for all i.

We get long exact sequences from chain complexes according to the following proposition.

Proposition 9.1. *Suppose we have a short exact sequence of chain complexes*

$$0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0.$$

Then there exist natural maps $\delta : H_i(C) \to H_{i-1}(A)$ producing a long exact sequence

$$\cdots \to H_i(A) \xrightarrow{f_i} H_i(B) \xrightarrow{g_i} H_i(C) \xrightarrow{\delta} H_{i-1}(A) \to \cdots .$$

Here, "naturality" of δ means that the maps we construct get us a functor from short exact sequences of chain complexes to long exact sequences. It doesn't seem like they test the precise meaning of "natural" here, so I wouldn't worry about it unless you're really curious.

We prove the proposition (ignoring naturality, since you won't be asked about it) here, as the construction of long exact sequences and of the connecting map does get tested. The key step, providing the construction of the connecting map, is important (and tested) enough to get its own proof.

Lemma 9.1 (Snake lemma)**.** *Suppose we have a commutative diagram with exact rows*

$$\begin{tikzcd}
 & A' \arrow[r, "\varphi'"] \arrow[d, "f"] & B' \arrow[r, "\psi'"] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r, "\varphi"] & B \arrow[r, "\psi"] & C. &
\end{tikzcd}$$

There is a map $\delta : \ker h \to \operatorname{coker} f$ fitting into an exact sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

Proof of snake lemma. We construct the map δ directly. Pick $x \in \ker h$. Exactness of the upper row at C' implies $B' \to C'$ is surjective, so $x = \psi'(y)$ for some $y \in B'$. By the commutativity of the diagram and the fact that $x \in \ker h$ we have $h(\psi'(y)) = \psi(g(y)) = h(x) = 0$, so $g(y) \in \ker \psi$ But exactness of the lower row at B then gives an element $z \in A$ with $\varphi(z) = g(y)$. This z is unique since exactness at A means φ is injective. We claim that sending x to the class $\overline{z}$ of z in $\operatorname{coker} f$ works for δ.

To see δ is well-defined, pick another $y' \in B'$ with $\psi'(y') = x$. Then $y - y' \in \ker \psi'$, so exactness at B' produces $w \in A'$ with $\varphi'(w) = y - y'$. Arguing as in the previous paragraph, there exists $z' \in A$ with $\varphi(z') = g(y')$, and the injectivity of φ along with the commutativity of the diagram then show that $f(w) = z - z'$. So both choices of y give the same class in $\operatorname{coker} f$, and δ is well-defined. (From here we can also show δ is a group homomorphism.)

Now we just need to check exactness. We have $\operatorname{im}(\ker g \to \ker h) \subseteq \ker \delta$, because for $x \in \ker h$ the image of $y \in \ker g$ we have $g(y) = 0$, and injectivity of $A \to B$ then implies $\delta(x) = 0 \in \operatorname{coker} f$. For $\ker \delta \subseteq \operatorname{im}(\ker g \to \ker h)$, take $x \in \ker h$ with $\delta(x) = 0$. We get $y \in B'$ with $\psi'(y) = x$ and $z \in A$ with $\varphi(z) = g(y)$. The fact that $\delta(x) = 0$ means $z \in \operatorname{im} f$, so there exists $w \in A'$ with $f(w) = z$. By the commutativity of the diagram we have $\varphi(f(w)) = g(\varphi'(w)) = g(y)$, so in particular $y - \varphi'(w) \in \ker g$. Then by exactness at B' we have

$$\psi'(y - \varphi'(w)) = \psi'(y) = x,$$

showing $x \in \operatorname{im}(\ker g \to \ker h)$ as required.

Finishing up, note that $\operatorname{im} \delta \subseteq \ker(\operatorname{coker} f \to \operatorname{coker} g)$ by construction of δ (in particular the fact that $\varphi(z) = g(y)$). To show $\ker(\operatorname{coker} f \to \operatorname{coker} g) \subseteq \operatorname{im} \delta$, pick $\overline{z} \in \ker(\operatorname{coker} f \to \operatorname{coker} g)$. This lifts to $z \in A$, and the fact that $\overline{z}$ is in the kernel of the map of cokernels means that $\varphi(z) \in \operatorname{im} g$. Pick $y \in B'$ with $g(y) = \varphi(z)$. Then $\psi'(y)$ satisfies $\delta(\psi'(y)) = \overline{z}$, and we are done. $\square$

For practical purposes, the most important part of the proof of the snake lemma is the first paragraph above, where we actually construct the map δ. This map is often called the *connecting map* or *boundary map*. You usually won't be asked to prove more of the snake lemma than that, but seeing the rest of the details is good practice with exact sequences.

Proof of proposition 9.1. The δs at each stage will be precisely the map constructed in the snake lemma, once we arrange the given complexes the right way. For all integers i, we have the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \ker d_A \arrow[r] \arrow[d] & \ker d_B \arrow[r] \arrow[d] & \ker d_C \arrow[d] & \\
0 \arrow[r] & A_i \arrow[r, "f_i"] \arrow[d, dashed, "d_A"] & B_i \arrow[r, "g_i"] \arrow[d, dashed, "d_B"] & C_i \arrow[r] \arrow[d, dashed, "d_C"] & 0 \\
0 \arrow[r] & A_{i-1} \arrow[r, "f_{i-1}"] \arrow[d] & B_{i-1} \arrow[r, "g_{i-1}"] \arrow[d] & C_{i-1} \arrow[r] \arrow[d] & 0 \\
& A_{i-1}/\mathrm{im}\,(d)_A \arrow[r] & B_{i-1}/\mathrm{im}\,(d)_B \arrow[r] & C_{i-1}/\mathrm{im}\,(d)_C \arrow[r] & 0
\end{tikzcd}$$

In particular, the top and bottom rows are exact. Moreover, we know that d_A, d_B, d_C commute with all f_i, g_i. This fact, combined with the fact that $d_A^2, d_B^2, d_C^2 = 0$, give another commutative diagram with exact rows

$$\begin{tikzcd}
& A_i/\operatorname{im} d_{i+1,A} \arrow[r] \arrow[d, dashed, "d_{i,A}"] & B_i/\operatorname{im} d_{i+1,B} \arrow[r] \arrow[d, dashed, "d_{i,B}"] & C_i/\operatorname{im} d_{i+1,C} \arrow[r] \arrow[d, dashed, "d_{i,C}"] & 0 \\
0 \arrow[r] & \ker d_{i-1,A} \arrow[r] & \ker d_{i-1,B} \arrow[r] & \ker d_{i-1,C}. &
\end{tikzcd}$$

We have $\ker d_{i,A} : A_i/\operatorname{im} d_{i+1,A} \to \ker d_{i-1,A} = H_i(A)$ and similar for B, C, while $\operatorname{coker} d_{i,A} : A_i/\operatorname{im} d_{i+1,A} \to \ker d_{i-1,A} = H_{i-1}(A)$ and similar for B, C. Applying the snake lemma to this diagram for all i then gives the desired long exact sequence. □

The following problems are mostly (if not entirely) focused on algebraic operations with chain complexes.

Spring 2013, #10 Let $A \subset X$ be a subspace of a topological space. Define the relative singular homology groups $H_p(X, A)$ and show that there is a long exact sequence

$$\cdots \to H_p(A) \to H_p(X) \to H_p(X, A) \to H_{p-1}(A) \to \cdots .$$

We note that on the level of chain complexes, we have $C_n(X, A) = C_n(X)/C_n(A)$. We note that this forms a chain complex where the boundary operator $\partial_n' : C_n(X, A) \to C_{n-1}(X, A)$ is given by $\partial_n'[\Delta] = [\partial_n \Delta]$. Note in particular that this is well defined, because if $[\Delta] = 0$, then we have $\Delta \in C_n(A)$, in which case $\partial_n \Delta \in C_{n-1}(A)$, so $[\partial_n \Delta] = 0$ as well. It is clear that we have a valid boundary operator since

$$\partial_{n-1}' \circ \partial_n'[\Delta] = [\partial_{n-1} \circ \partial_n \Delta] = [0] = 0$$

Therefore, we can define the relative homology as

$$H_n(X, A) = \ker(\partial_n) / \operatorname{Im}(\partial_{n+1})$$

Now, we claim that $0 \to C(A) \to C(X) \to C(X,A) \to 0$ forms a short exact sequence of chain complexes. To see this we first note at for each $n, 0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$ is a short exact sequence of abelian groups. Denote the maps as $0 \to C(A) \xrightarrow{i} C(X) \xrightarrow{j} C(X,A) \to 0$. Where i is the inclusion and j is the quotient map. It is clear that $i \circ \partial_n = \partial_n \circ i$. We wish now to show that $j \circ \partial_n = \partial_n \circ j$. Indeed, we see that $j \circ \partial_n'[\Delta] = j(\partial_n \Delta) = [\partial_n \Delta]$, while $\partial_n \circ j(\Delta) = \partial_n'[\Delta] = [\partial_n \Delta]$, as desired.

Since we have a short exact sequence of chain complexes, we then have a long exact sequence of homology groups (via a lot of diagram chasing), as desired. ∎

Fall 2019, #9

(a) If
$$0 \to A \to B \to C \to 0$$
is a short exact sequence of chain complexes, show how to get the boundary map in the associated long exact sequence.

(b) Compute the boundary map when the short exact sequence is the result of tensoring the chain complex
$$\cdots \to 0 \to \mathbb{Z} \xrightarrow{5} \mathbb{Z} \to 0 \to \cdots$$
with the short exact sequence
$$0 \to \mathbb{Z}/5 \xrightarrow{5} \mathbb{Z}/25 \to \mathbb{Z}/5 \to 0.$$

(a) Let $i : A \to B$ and $j : B \to C$ be the maps between the chain complexes. We construct the map $H_n(C) \to H_{n-1}(A)$. First, take $[c] \in H_n(C)$ Since c is cycle, we have $\partial c = 0$. From short exactness, j is surjective, so there is $b \in B_n$ such that $b \mapsto c$. Now, we note that $\partial j(b) = j\partial b = 0$, so $j\partial b = 0$, meaning that there is $a \in A_{n-1}$ such that $i(a) = j(\partial b)$. This $[a]$ is what $[c]$ is sent to. Now, we note that if we took another b' such that $j(b') = c$, then we see that $j(b - b') = 0$. This means that there is $\tilde{a} \in A_n$ such that $i(\tilde{a}) = b - b'$. This means that for the $a' \in A_{n-1}$ such that $i(a') = j(\partial b)$, we would have $i(a - a') = i(a) - i(a') = \partial(b) - \partial(b') = \partial(b - b') = \partial i(\tilde{a}) = i\partial(\tilde{a})$; since i is injective, this means $a - a' = \partial \tilde{a}$, so a and $\tilde{a}$ are in the same homology.

Finally, we note that if we took $c + \partial\tilde{c}$, then we notice the following: there is $\tilde{b}$ in B_{n+1} that maps to $\tilde{c} \in C_{n+1}$, in which case, $b + \partial\tilde{b}$ mapsto to $c + \partial\tilde{c}$. Applying this ∂ to $b + \partial\tilde{b}$ yields ∂b, since $\partial \circ \partial \equiv 0$. Thus, the $[a]$ obtained is still the same.

(b) Tensoring the two short exact sequences together gives the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_5 \arrow[r, "\times 5"] \arrow[d, "0"] & \mathbb{Z}_{25} \arrow[r] \arrow[d, "\times 5"] & \mathbb{Z}_5 \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & \mathbb{Z}_5 \arrow[r, "\times 5"] & \mathbb{Z}_{25} \arrow[r] & \mathbb{Z}_5 \arrow[r] & 0
\end{tikzcd}$$

We note that the homology group on the top right is $\mathbb{Z}/5$. If we take $1 \in \mathbb{Z}/5$, we note that $1 \in \mathbb{Z}/25$ maps to it. Pushing it down, we note that we get $5 \in \mathbb{Z}/25$. We note that this gets mapped to by $1 \in \mathbb{Z}/5$, which is the output of the boundary map. So, in this case, we see that the boundary map is the identity on $\mathbb{Z}/5$.

$\blacksquare$

Fall 2022, #6 Let C_* be a chain complex of free abelian groups. Let $A_* = C_* \otimes \mathbb{Z}/p$ and let $B_* = C_* \otimes \mathbb{Z}/p^2$ be the chain complexes we get by tensoring C_* degreewise with $\mathbb{Z}/p$ and $\mathbb{Z}/p^2$, respectively.

(a) Show that we have a short exact sequence of chain complexes

$$0 \to A_* \to B_* \to A_* \to 0$$

induced by the corresponding sequence of chain complexes

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

(b) Show how to define a Bockstein natural transformation

$$\beta : H_k(A_*) \to H_{k-1}(A_*)$$

producing a long exact sequence

$$\cdots \to H_k(A_*) \to H_k(B_*) \to H_k(A_*) \xrightarrow{\beta} H_{k-1}(A_*) \to \cdots$$

associated to the short exact sequence of part (a).

(c) Show that if x and y are elements such that $d(x) = py$, then

$$\beta(\overline{x}) = \overline{y},$$

where the bars indicate the reduction modulo p of the corresponding classes.

(d) Show conversely that given an element $\overline{x} \in H_k(A_*)$, if $\beta(\overline{x}) = 0$, then we can find elements $x, y \in C_*$ such that x reduces to $\overline{x}$ modulo p and $d(x) \equiv p^2 y \pmod{p^3}$.

9.2 Homology for Cursed (and Not-So-Cursed) Spaces: The Mayer-Vietoris Sequence

The most common type of long exact sequence problem asks you to compute the homology of a concrete space. There is a particular type of space that lends itself to these methods: generally, these will be quotients of products of spaces you know well. The Mayer-Vietoris sequence lets us compute (co)homology for these spaces using a nice cover for the space, typically picking two subsets homotopy equivalent to spaces whose homology we know, whose intersection also has familiar homology.

Theorem 9.2 (Mayer-Vietoris for singular homology)**.** *Let X be a topological space and A, B two subspaces whose interiors cover X. Denote by i (resp. j) the inclusion of $A \cap B$ into A (resp. B), and denote by k (resp. ℓ) the inclusion of A (resp. B) into X. Then there is a long exact sequence*

$$\cdots \to H_{n+1}(X) \xrightarrow{\delta} H_n(A \cap B) \xrightarrow{\binom{i_*}{j_*}} H_n(A) \oplus H_n(B) \xrightarrow{k_* - \ell_*} H_n(X) \xrightarrow{\delta} H_{n-1}(A \cap B) \to \cdots.$$

Moreover, if A and B have non-empty intersection, there is a Mayer-Vietoris exact sequence for reduced homology given by putting tildes above every H in the standard Mayer-Vietoris sequence.

With Mayer-Vietoris it is often helpful to know the actual maps on homology, which is why we state the theorem with the maps included. The map δ comes from the snake lemma as follows: for

$i \geqslant 0$, denote by $C_i(A+B)$ the subgroup of the chain group $C_i(X)$ consisting of sums of chains in A and chains in B. Barycentric subdivision shows that the inclusion $C_i(A+B) \hookrightarrow C_i(X)$ induces isomorphisms on homology. We obtain a short exact sequence

$$0 \to C_i(A \cap B) \to C_i(A) \oplus C_i(B) \to C_i(A+B) \to 0$$

giving a short exact sequence of chain complexes, after which the formalism of the previous section produces the long exact sequence with boundary map as in the snake lemma.

The following problem is both good practice with Mayer-Vietoris and is common enough in its own right to be worth knowing.

Spring 2014, #10; Spring 2016, #9; Fall 2018, #9; Fall 2020, #6; Spring 2022, #8 Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point, and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

Take A to be the image of $X \times [0, .55)$ and B to be the image of $X \times (.45, 1]$ in $S(X)$. Then A and B are contractible and $A \cap B$ deformation retracts onto a copy of X. The Mayer-Vietoris sequence for reduced homology reads

$$\cdots \to \tilde{H}_{i+1}(A) \oplus \tilde{H}_{i+1}(B) \to \tilde{H}_{i+1}(A \cup B) \to \tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \cdots$$

which in this situation becomes

$$\cdots \to 0 \to \tilde{H}_{i+1}(S(X)) \to \tilde{H}_i(X) \to 0 \to \cdots.$$

This portion of the exact sequence tells us that $\tilde{H}_{i+1}(S(X)) \cong \tilde{H}_i(X)$ for all $i \geqslant 1$. Since $S(X)$ is path-connected we have $\tilde{H}_0(S(X)) = 0$, and so the portion of the sequence going from degree 1 to degree 0 reads

$$0 \to \tilde{H}_1(S(X)) \to \tilde{H}_0(X) \to 0.$$

In particular, we have $\tilde{H}_{k+1}(S(X)) \cong \tilde{H}_k(X)$ for all k. ∎

While this is a good example to know, it's not the hardest one you might see. For instance, we did not need to do any "exact sequence sudoku" since all of the maps were isomorphisms. In tandem with this point is that we didn't have to think about what the maps in the sequence are doing. The next several examples offer plenty of practice with computing homology of spaces cooked up from ones we know.

Fall 2013, #10 Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the group of quaternions, with relations $i^2 = j^2 = -1$, $ij = -ji = k$. The multiplicative group $\mathbb{H}^* = \mathbb{H} - \{0\}$ by left multiplication. The quotient $\mathbb{HP}^{n-1} = (\mathbb{H}^n - \{0\})/\mathbb{H}^*$ is called the quaternionic projective space. Calculate its homology groups. (It is easiest to do this with cellular homology, but it's good to know how to calculate (co)homology of projective spaces using Mayer-Vietoris and induction, also.)

Fall 2014, #7 A compact surface of genus g, smoothly embedded in $\mathbb{R}^3$, bounds a compact region called a *handlebody* H.

(a) Prove that two copies of H glued together along their boundaries by the identity map produces a closed topological 3-manifold M.
(b) Compute the homology of M.
(c) Compute the relative homology of (M, H), where H is one of the two copies.

(a) We note that for the interior points of M, it correponds to an interior point of H, and therefore they have a neighborhood that is homeomorphic to $\mathbb{R}^3$ since H is a manifold. The boundary points of M correspond to the points which are a boundary point of H, and thus have a neighborhood that is homeomorphic to the closed halfspace; since we are identifying the two copies of H by the boundary, it follows that the neighborhood we choose gets identified along the boundary, which is homeomorphic to the halfspace identified along the boundary, which is precise $\mathbb{R}^3$. We have compactness by the fact that we are taking the quotient of a compact space.

(b) Let A be the interior of $H \times \{0,1\}$, and B be a thickening of the boundary. As such, we note that A deformation retracts onto three copies of the wedge of g circles, and B deformation retracts onto ∂H, the compact oriented genus g surface. We also note that $A \cap B$ deformation retracts onto three copies of ∂H. For A, we note that $H_2(A) = 0, H_1(A) = \mathbb{Z}^{2g}$, generated by a_i^j, for $j = 0,1$, and $i = 1, \cdots, g$, and $H_0(A) = \mathbb{Z}^2$. For B, we note that $H_2(B) = \mathbb{Z}$, $H_1(B) = \mathbb{Z}^{2g}$, generated by $a_1, b_1, \cdots, a_g, b_g$, and $H_0(B) = 0$. As for $A \cap B$, we have $H_2(A \cap B) = \mathbb{Z}^2, H_1(A \cap B) = \mathbb{Z}^{4g}$, generated by a_i^j and b_i^j, for $i = 0,1$, and $i = 1, \cdots, g$. and $H_0(A \cap B) = \mathbb{Z}^3$. From this, we have the following Mayer Vietoris sequence:

$$\begin{aligned} 0 \to H_3(X) \to H_2(A \cap B) &\to H_2(A) \oplus H_2(B) \to H_2(X) \\ &\to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(X) \\ &\to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0 \end{aligned}$$

We note that X is clearly path connected, so $H_0(X) = \mathbb{Z}$. We also note that $H_0(A \cap B)$ injects into $H_0(A) \oplus H_0(B)$, since the generators for each path component of $A \cap B$ maps to the generator of the corresponding path component in $H_0(A)$. Thus, we see that the last row is exact, in which case we get

$$\begin{aligned} 0 \to H_3(X) \to H_2(A \cap B) &\to H_2(A) \oplus H_2(B) \to H_2(X) \\ &\to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(X) \to 0 \end{aligned}$$

Now, we note that for the map $H_2(A \cap B)) \to H_2(A) \oplus H_2(B)$, we have $(1, -1) \mapsto 0$, since $H_2(A) = 0$ and the image of each path component of $A \cap B \to B$ is homotopy equivalent to B. Thus, we see that the kernel of this map is isomorphic to $\mathbb{Z}$, in which case we see that the image of the map $H_3(X) \to H_2(A \cap B)$ has as its image isomorphic to $\mathbb{Z}$; indeed, since this map is injective, $H_3(X) \cong \mathbb{Z}$. Now, we note that since $H_2(A \cap B) \cong \mathbb{Z}^2$ and $H_2(A \cap B) \to H_2(B)$ is surjective, we see that $0 \to H_3(X) \to H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to 0$ is a short exact sequence. As such, the following sequence is then exact:

$$0 \to H_2(X) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B) \to H_1(X) \to 0.$$

We note that the map $H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ maps $a_i^j \mapsto \left(a_i^j, a_i\right)$ and $b_i^j \mapsto (0, b_i)$. From this, we see that the kernel of this map is generated by $b_i^1 - b_i^2$, for $i = 1, \cdots, n$. This is isomorphic to $\mathbb{Z}^{2g}$ generators, from which we see that the image of the map $H_2(X) \to H_1(A \cap B)$ is too, and since it's injective, $H_2(X) \cong \mathbb{Z}^g$. Finally, we notice that the map $H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ has as its image generated by $\left(a_i^j, a_i\right)$ and $(0, b_i)$, which has dimension $3g$, which is the kernel of the next map. Moreover, we can extend this by $(0, a_i)$, which gives a generating set for $H_1(A) \oplus H_1(B)$. Thus, we see since $H_1(A) \oplus H_1(B) \to H_1(X)$ is surjective, we have $H_1(X) \cong H_1(A) \oplus H_1(B) / \ker\left(H_1(A) \oplus H_1(B) \to H_1(X)\right)$, which is generated by the equivalence classes of $(0, a_i)$. It thus follows that $H_1(X) \cong \mathbb{Z}^g$. We note $H_k(X) = 0$ for $k > 3$, since X is the quotient of a 3-manifold.

■

Fall 2016, #10 If $f : X \to X$ is a self-map, then the mapping torus of f is the quotient

$$T_f := (X \times [0,1])/((x,0) \sim (f(x),1)).$$

For $n \in \mathbb{Z}$, let f_n be a degree n map $S^3 \to S^3$. Compute the homology groups of T_{f_n}.

Fall 2017, #9 A compact surface (without boundary) of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact 3-dimensional region called a *handlebody* H. Let $X = (H \times \{0,1,2\})/\sim$, where $(x,i) \sim (x,j)$ for all $x \in \partial H$ and $i, j \in \{0,1,2\}$. Compute the homology of X.

Spring 2018, #7 Let M, N be smooth, connected, orientable n-manifolds for $n \geqslant 3$, and let $M\#N$ denote their connected sum.

(a) Compute the fundamental group of $M\#N$ in terms of those of M, N. You may assume the base point is on the boundary sphere along which we glue M and N.

(b) Compute the homology groups of $M\#N$. You may use without proof that $H_n(-;\mathbb{Z})$ of a connected orientable n-manifold is always isomorphic to $\mathbb{Z}$.

(c) For part (a), what changes if $n = 2$? Use this to describe the fundamental groups of orientable surfaces.

(This is more of a van Kampen problem than a Mayer-Vietoris problem, but why not do a bit of both :))

Fall 2019, #4 Let $X = S^1 \times S^1$ and let Y be the quotient of $X \times [0,1]$ by the relation

$$((x,y),0) \sim ((y,x),1).$$

Compute $H_*(Y;\mathbb{Z})$.

Spring 2020, #10 Let D^2 be the unit disk in $\mathbb{C}$, and let $S^1 = \partial D^2$. Let $X = D^2 \times S^1 \times \{0,1\}/\sim$ where

$$(x,y,0) \sim (xy^5, y, 1)$$

for all $x, y \in S^1$. Compute the homology groups of X.

Fall 2021, #9 Let X be the quotient of the space $\{0,1,2\} \times S^1 \times D^2$ by the relation

$$(0,z_1,z_2) \sim (1,z_1,z_2) \sim (2,z_1,z_2) \quad \forall z_1, z_2 \in S^1.$$

Compute the homology groups $H_n(X;\mathbb{Z})$ for all n.

Spring 2023, #10 Consider the CW-complexes $A = S^n \vee S^n$, $X = S^n \times S^n$, and $B = S^n \times [0,1]/* \times [0,1]$, where $*$ is the basepoint of S^n. There are inclusions $A \hookrightarrow X$ given by the pairs of points where at least one is the basepoint, and $A \hookrightarrow B$ which takes one S^n to $S^n \times 0$ and the other to $S^n \times 1$. Compute the homology of
$$Y = X \cup_A B.$$

9.2.1 Mayer-Vietoris for Cohomology

Mayer-Vietoris also works for cohomology, but it goes the opposite way. For singular cohomology, the sequence looks like

$$\cdots \to H^i(X) \to H^n(A) \oplus H^i(B) \to H^i(A \cap B) \xrightarrow{\delta} H^{i+1}(X) \to \cdots$$

with dimension-preserving maps given by restriction. We give a precise statement for de Rham cohomology since it is slightly different and we can again describe the maps explicitly.

Theorem 9.3 (Mayer-Vietoris, de Rham cohomology)**.** *Let M be a smooth manifold, and let $U, V \subset M$ be open subsets which cover M. Let i (resp. j) denote the inclusion $U \cap V \hookrightarrow U$ (resp. $U \cap V \hookrightarrow V$), and let k (resp. ℓ) denote the inclusion $U \hookrightarrow M$ (resp. $V \hookrightarrow M$). Then there are natural maps $\delta : H^i_{\mathrm{dR}}(U \cap V) \to H^{i+1}_{\mathrm{dR}}(M)$ fitting into a long exact sequence*

$$\cdots \to H^i_{\mathrm{dR}}(M) \xrightarrow{\binom{k^*}{\ell^*}} H^i(U) \oplus H^i_{\mathrm{dR}}(V) \xrightarrow{i^* - j^*} H^i_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta} H^{i+1}_{\mathrm{dR}}(M) \to \cdots.$$

As far as I can tell, Mayer-Vietoris for cohomology is very rare compared to for homology. That said, the strategies are mostly the same: pick your cover wisely using deformation retracts, know the (co)homology of common spaces, and study the exact sequence you get.

Spring 2016, #6 Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus with coordinates $(x, y) \in \mathbb{R}^2$, and let $p \in T^2$.
(a) Compute the de Rham cohomology of the punctured torus $T^2 - \{p\}$.
(b) Is the volume form $\omega = dx \wedge dy$ exact on $T^2 - \{p\}$?

It is also possible to use Mayer-Vietoris to compare Euler characteristics of spaces, using the fact that the alternating sum of ranks / dimensions in an exact sequence is zero. The following problem, which has already appeared a couple times on this document, provides the key example on this front.

Spring 2016, #4, Spring 2022, #9 Let M be a compact odd-dimensional maniofld with nonempty boundary ∂M. Show that the Euler characteristics of M and ∂M are related by

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

9.3 Relative Homology and the Long Exact Sequence of a Pair

The other major long exact sequences that come into play frequently on the qual are the long exact sequence for relative homology, and the long exact sequence of a good pair. Recall that a space X and subspace A form a good pair if A is nonempty, closed, and is the deformation retract of some neighborhood in X. These sequences are useful for computing relative homology and for computing the homology of a quotient X/A where (X, A) is a good pair.

Definition 9.5. *Let X be a topological space and $A \subset X$ a subspace. Since the boundary map for the singular complex takes i-chains in A to $i-1$ chains in A, there is a well-defined boundary map on the quotients $C_i(X)/C_i(A)$. The* relative homology groups *$H_i(X, A)$ are then given by the homology of the resulting complex.*

We have a short exact sequence of complexes

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0$$

where $C_i(X, A) := C_i(X)/C_i(A)$ for all i. The algebraic machinery of the first section then gives the following.

Theorem 9.4. *Let X be a topological space and $A \subset X$ a subspace. There are natural maps $\delta : H_i(X, A) \to H_{i-1}(A)$ for all i, producing a long exact sequence*

$$\cdots \to H_i(A) \xrightarrow{i_*} H_i(X) \to H_i(X, A) \xrightarrow{\delta} H_{i-1}(A) \to \cdots .$$

The connecting map δ sends the class of a relative cycle α in $H_i(X, A)$ to the class of $\partial\alpha$ in $H_{i-1}(A)$. Moreover, when A is nonempty, we get an exactly analogous exact sequence for reduced homology.

We can already do some calculations with this sequence!

Spring 2017, #7 Let $X = S^1 \times D^2$ with boundary $\partial X = S^1 \times S^1$. Compute the relative homology groups $H_k(X, \partial X; \mathbb{Z})$ for all k.

Spring 2020, #7 Prove that the relative homology groups $H_k(X, x)$ for different choices of basepoint x can be naturally identified with each other. That is, for every $k \geqslant 0$, every space X, and all pairs of points $x, y \in X$ (not necessarily in the same connected component), construct isomorphisms $\eta^X_{x,y} : H_k(X, x) \to H_k(X, y)$ satisfying
(a) $\eta^X_{x,x} = \text{id}$ for all $x \in X$;
(b) $\eta^X_{y,z} \circ \eta^X_{x,y} = \eta^X_{x,z}$ for all $x, y, z \in X$;
(c) $f_* \circ \eta^X_{x,y} = \eta^Y_{f(x),f(y)} \circ f_*$ for all $x, y \in X$ and all continuous maps $f : X \to Y$.
(Hint: consider doing the $k \geqslant 1$ case first.)

One technique for simplifying relative homology calculations is to use the excision theorem. This theorem says that if you start with a subspace $A \subset X$ and remove a small enough $Z \subset A$ from A and Z, then the relative homology does not change. While this theorem is useful, for example, to prove the exact sequence for reduced homology of a good pair, it doesn't directly turn up on the qual much (a search for the word "excision" in Jerry's notes brought back nothing...). So, we simply state the theorem here, then move on to the sequence for good pairs.

Theorem 9.5 (Excision)**.** *Given subspaces $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, the inclusion $(X - Z, A - Z) \hookrightarrow (X, A)$ induces isomorphisms $H_i(X - Z, A - Z) \to H_i(X, A)$ for all i. Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms $H_i(B, A \cap B) \to H_i(X, A)$ for all i.*

Theorem 9.6 (Exact sequence for good pairs)**.** *If (X, A) is a good pair (i.e. A is closed in X and is a deformation retract of some neighborhood in X), then there is an exact sequence*

$$\to \tilde{H}_i(A) \xrightarrow{i_*} \tilde{H}_i(X) \xrightarrow{j_*} \tilde{H}_i(X/A) \xrightarrow{\delta} \tilde{H}_{i-1}(A) \to \dots,$$

where i is the inclusion $A \hookrightarrow X$ and j is the quotient map $X \to X/A$. Moreover, the quotient map $q : (X, A) \to (X/A, A/A)$ induces isomorphisms $H_i(X, A) \simeq \tilde{H}_i(X/A)$ for all i.

The main thing I've seen that uses the sequence for a good pair is the following problem:

> **Fall 2019, #2, Spring 2023, #9** Compute $H_*(\mathbb{RP}^{n+m}/\mathbb{RP}^n; \mathbb{Z})$ as a function of n and m. Here we are viewing $\mathbb{RP}^n \subset \mathbb{RP}^{n+m}$ induced from the inclusion
>
> $$\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^{n+m+1},$$
> $$(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{n+1}, 0, \dots, 0).$$

(This is written slightly differently but the idea is there)

First note that $(\mathbb{R}P^n, \mathbb{R}P^m)$ is a good pair, so $\tilde{H}_i(\mathbb{R}P^n/\mathbb{R}P^m) = H_i(\mathbb{R}P^n, \mathbb{R}P^m)$.

We have $H_i(\mathbb{R}P^m) = 0$ for all $i > m$, and $H_i(\mathbb{R}P^m) \to H_i(\mathbb{R}P^n)$ is an isomorphism for $i < m$. From this, it follows that

$$H_i(\mathbb{R}P^n, \mathbb{R}P^m) \cong H_i(\mathbb{R}P^n) \quad \text{for } i > m+1$$

and $H_i(\mathbb{R}P^n, \mathbb{R}P^m) = 0$ for $i < m$. For $i = m, m+1$, we have the exact sequence

$$0 \to H_{m+1}(\mathbb{R}P^n) \to H_{m+1}(\mathbb{R}P^n, \mathbb{R}P^m) \to H_m(\mathbb{R}P^m) \to H_m(\mathbb{R}P^n) \to H_m(\mathbb{R}P^n, \mathbb{R}P^m) \to 0.$$

There are two cases.

Suppose m is even. Then, $H_m(\mathbb{R}P^m) = 0$, thus $H_i(\mathbb{R}P^n, \mathbb{R}P^m) \cong H_i(\mathbb{R}P^n)$ for $i = m, m+1$.

Suppose m is odd. Then, $H_m(\mathbb{R}P^m) \cong \mathbb{Z}$ and $H_m(\mathbb{R}P^n) = \mathbb{Z}/2$, and our sequence takes the form

$$0 \to H_{m+1}(\mathbb{R}P^n, \mathbb{R}P^m) \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2 \to H_m(\mathbb{R}P^n, \mathbb{R}P^m) \to 0$$

where ϕ is the map induced by inclusion. When we consider the inclusion $\mathbb{R}P^m \subset \mathbb{R}P^n$, the top m-cell of the subspace gets an $(m+1)$-cell attached to it in the larger space via a map of degree 2, and from the cellular chain complex you see that this m-cell that generates $H_m(\mathbb{R}P^m) \cong \mathbb{Z}$ also generates $H_m(\mathbb{R}P^n) \cong \mathbb{Z}/2$. In other words ϕ is surjective, and therefore $H_m(\mathbb{R}P^n, \mathbb{R}P^m) = 0$ and $H_{m+1}(\mathbb{R}P^n, \mathbb{R}P^m) \cong \ker(\phi) \cong \mathbb{Z}$.

∎

There is also the following interesting problem, to keep you on your toes with hypotheses:

> **Spring 2019, #7** Let $X = [0, 1]$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}, n \geqslant 1\}$. Show that $H_1(X, A)$ is not isomorphic to $H_1(X/A)$.

There are a few more problems below.

9.4 Problems

Spring 2015, #9 Let X, Y be topological spaces and let $f, g : X \to Y$ be two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \sqcup Y$ by identifying $(x,0) \sim f(x)$ and $(x,1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form

$$\cdots \to H_i(X) \xrightarrow{a} H_i(Y) \xrightarrow{b} H_i(Z) \xrightarrow{c} H_{i-1}(X) \to \cdots$$

and describe the maps a, b, c.

Fall 2015, #9 Given a continuous map $f : X \to Y$ between topological spaces, define

$$C_f = ((X \times [0,1]) \sqcup Y) / \sim,$$

where $(x,1) \sim f(x)$ for all $x \in X$ and $(x,0) \sim (x',0)$ for all $x, x' \in X$. Show that there is a long exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H_{i+1}}(C_f) \to H_i(X) \xrightarrow{f_*} \to \cdots,$$

where f_* is the map on homology induced from f and $\tilde{H}_i$ denotes the ith reduced homology group.

Spring 2021, #4 Let $\Delta_n^{(k)}$ be the k-dimensional skeleton of the n-simplex Δ_n. Calculate the reduced homology groups $\tilde{H}_i(\Delta_n^{(k)})$ for all values of i, k, n.

Fall 2022, #9 The space $S^1 \times S^1$ is the mapping cone of the map

$$[a,b] : S^1 \to S^1 \vee S^1,$$

representing the commutator of the inclusion of the left summand $a : S^1 \to S^1 \vee S^1$ and the inclusion of the right summand $b : S^1 \to S^1 \vee S^1$. Use this and the long exact sequence to compute the homology.

10 Suspensions and Mapping Cylinders/Cones/Tori

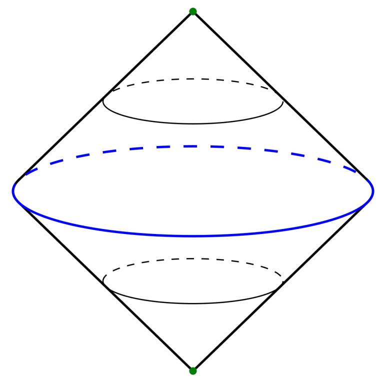
Here we outline specific constructions that have appeared on the qual and which are relevant to homology. They are generally not as ubiquitous as the long exact sequences described in the last section, but it is good to know how to work with them. In particular, suspensions interact with homology and degree in a way that can provide quick solutions to some problems, and the mapping stuff tends to produce some of the more difficult exact sequence problems.

10.1 Suspensions

This section is largely drawn from Sam's qual prep document, albeit more condensed and with more focus on getting the most useful facts and practice problems written down. We start with the definition.

Definition 10.1. *The* suspension $S(X)$ *of a space* X *is the quotient space* $(X \times [0,1])/\sim$, *where* $(x,1) \sim (y,1)$ *and* $(x,0) \sim (y,0)$ *for all* $x, y \in X$.

Informally, we form $S(X)$ by "suspending" X between two points, as shown in the following classic picture of $S(S^1) \simeq S^2$:



The most useful fact about suspensions for qual purposes, which often arises as a qual problem itself, is the following characterization of the homology of $S(X)$ (copied from the section on Mayer-Vietoris).

Spring 2014, #10; Spring 2016, #9; Fall 2018, #9; Fall 2020, #6; Spring 2022, #8 Let X be a topological space. Define the suspension $S(X)$ to be the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point, and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

Take A to be the image of $X \times [0, .55)$ and B to be the image of $X \times (.45, 1]$ in $S(X)$. Then A and B are contractible and $A \cap B$ deformation retracts onto a copy of X. The Mayer-Vietoris sequence for reduced homology reads

$$\cdots \to \tilde{H}_{i+1}(A) \oplus \tilde{H}_{i+1}(B) \to \tilde{H}_{i+1}(A \cup B) \to \tilde{H}_i(A \cap B) \to \tilde{H}_i(A) \oplus \tilde{H}_i(B) \to \cdots$$

which in this situation becomes

$$\cdots \to 0 \to \tilde{H}_{i+1}(S(X)) \to \tilde{H}_i(X) \to 0 \to \cdots .$$

This portion of the exact sequence tells us that $\tilde{H}_{i+1}(S(X)) \cong \tilde{H}_i(X)$ for all $i \geqslant 1$. Since $S(X)$ is path-connected we have $\tilde{H}_0(S(X)) = 0$, and so the portion of the sequence going from degree 1 to degree 0 reads

$$0 \to \tilde{H}_1(S(X)) \to \tilde{H}_0(X) \to 0.$$

In particular, we have $\tilde{H}_{k+1}(S(X)) \cong \tilde{H}_k(X)$ for all k. ∎

A punchy summary of the problem above is "suspension raises the degree of homology." A couple consequences of this fact include:

- For $k \geqslant 2$, we have $H_k(S(X)) \cong H_{k-1}(X)$.
- For $k = 1$ we get $H_1(S(X)) \cong \tilde{H}_0(X) \cong \mathbb{Z}^{k-1}$, where k is the number of path components of X.

- More an observation than a consequence, but we always have $H_0(S(X)) = \mathbb{Z}$ because $S(X)$ is always path connected: from any point one can walk up to one of the suspension points and back down to where they need to go.

This interaction of the suspension with homology is useful for some problems, e.g. to get spaces with prescribed homology groups.

Fall 2013, #8 Let $n > 0$ be an integer and A a finitely presented abelian group. Show that there is a space X with $H_n(X) = A$.

As an extension of this problem, note that we can get prescribed homology in finitely many degrees by repeatedly applying the method of this problem, taking wedge sums, and possibly taking disjoint union with some isolated points to modify H_0.

If (X, x_0) is a pointed space with chosen point x_0, the space $S(X)$ has a whole interval of points coming from x_0. In particular $S(X)$ does not have a natural choice of base point. We can get around this issue using *reduced suspensions*:

Definition 10.2. *If (X, x_0) is a pointed space (alternatively, if X is any space and we pick any $x_0 \in X$), the* reduced suspension ΣX *is the space* $(X \times [0,1])/\left((X \times \{0,1\}) \cup (\{x_0\} \times [0,1])\right)$.

Reduced suspension interacts with homology in a manner simiilar to the unreduced suspension.

Spring 2016, #9 Let $p \in X$ and ΣX be the reduced suspension of X: that is, taking $X \times [0,1]$ and collapsing $X \times \{0,1\} \cup p \times [0,1]$ into a point. Describe the relation between the homology groups of X and ΣX.

We first begin with the same argument as in 10.1 to get $\widetilde{H}_{k+1}(SX) \cong \widetilde{H}_k(X)$ for all k. Then, considering $\Sigma X = SX/(\{p\} \times [0,1])$, we note we have the LES, as $(SX, \{p\} \times [0,1])$ is a good pair,

$$\cdots \to \widetilde{H}_k(\{p\} \times [0,1]) \to \widetilde{H}_k(SX) \to \widetilde{H}_k(\Sigma X) \to \dots$$

As $\{p\} \times [0,1]$ is contractible, this implies

$$\widetilde{H}_{k+1}(\Sigma X) \cong \widetilde{H}_{k+1}(SX) \cong \widetilde{H}_k(X),$$

for all k. ■

It is worth noting that suspension is a *functor*: not only can we suspend spaces, we can also suspend continuous maps, and this suspension operation respects composition of maps. Explicitly, let $f : X \to Y$ be a continuous map of spaces. We get a continuous map $f \times \mathrm{id} : X \times [0,1] \to Y \times [0,1]$ sending (x, t) to $(f(x), t)$. Taking the quotient giving the suspension on the codomain gives $f \times \mathrm{id} : X \times [0,1] \to S(Y)$, and since this map is constant on the fibers of $\pi : X \times [0,1] \to S(X)$, we obtain a well-defined continuous function $S(f) : S(X) \to S(Y)$ given by the formula above.

Since we can iterate functors, we can also form iterated suspensions $S^n(X)$ and reduced suspensions $\Sigma^n(X)$.

Fall 2022, #10 Let $f : X \to Y$ be a continuous map of pointed spaces. Let $\Sigma^n f : \Sigma^n X \to \Sigma^n Y$ be the nth reduced suspension of f. Show that if for some n, $\Sigma^n f$ induces the trivial map on reduced homology, then it does for all n.

A final fact on suspensions is that the preserve degrees: if $f : S^n \to S^n$ is a degree k map, then $S(f) : S^{n+1} \to S^{n+1}$ also has degree k. See Subsection 12.1, the subsection on degrees in algebraic topology, for a proof.

10.2 Mapping Cylinders/Cones/Tori

These constructions don't seem to appear very often on the qual, but they can be tricky when they do. Oftentimes you will be given the construction of one of these objects and be asked to derive a long exact sequence relating the homology of the construction to those of the constituent spaces and the map used to glue.

First, a quick definition and exposition.

Definition 10.3. *Let $f : X \to Y$ be a continuous map of spaces. The* mapping cylinder of f *is the space M_f formed by gluing $X \times [0,1]$ to Y by identifying $X \times \{1\}$ with $f(X)$. In symbols, we have $M_f = ((X \times [0,1]) \coprod Y)/\sim$ where $(x,1) \sim f(x)$ for all $x \in X$.*

The idea behind mapping cylinders is that it allows you to "treat all maps as inclusions" from the point of view of homotopy theory. What does this mean? Notice that M_f deformation retracts onto Y by sliding the copies of X down. So $M_f \simeq Y$ (homotopy equivalence). This deformation retraction also gives a homotopy from the inclusion $X \hookrightarrow X \times \{0\} \subset M_f$ to $X \xrightarrow{f} f(X) \subset M_f$, showing that the inclusion $X \hookrightarrow M_f$ and f itself induce the same map on homology. Therefore, although f might not be injective, we can replace it by the inclusion $X \hookrightarrow M_f$ which has the same homotopical behavior.

This idea is useful for deriving an exact sequence in homology for the *mapping cone* $C_f = M_f/(X \times \{0\})$.

> **Fall 2015, #9** Given a continuous map $f : X \to Y$ between topological spaces, define
>
> $$C_f = \left(X \times [0,1] \coprod Y\right)/\sim,$$
>
> where $(x,1) \sim f(x)$ for all $x \in X$ and $(x,0) \sim (x',0)$ for all $x, x' \in X$. Here $\coprod$ is the disjoint union. Show that there is a long exact sequence
>
> $$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$
>
> where f_* is the map on homology induced from f and $\tilde{H}_i$ denotes the ith reduced homology group.

You can do this using Mayer-Vietoris or the sequence of pairs. Here we will use the sequence of pairs.

Let M_f be the mapping cone of f and $A \subset M_f$ be $X \times \{0\}$. Then $C_f = M_f/A$ and (M_f, A) form a good pair, since A is closed in M_f and A has a neighborhood (say $X \times [0, 0.5)$) which deformation retracts onto it. So $H_i(M_f, A) \cong \tilde{H}_i(M_f/A) = \tilde{H}_i(C_f)$ for all i, and the sequence of pairs reads

$$\cdots \to H_{i+1}(A) \xrightarrow{\iota_*} H_{i+1}(M_f) \to \tilde{H}_{i+1}(C_f) \to H_i(A) \xrightarrow{\iota_*} H_i(M_f) \to \cdots,$$

where $\iota : A \hookrightarrow M_f$ is the inclusion.

This sequence already looks a lot like what we want; we just need to swap out that A and M_f terms while keeping track of what happens to ι upon making these identifications. Since the natural map $X \hookrightarrow A$ is a homeomorphism, we have $H_i(X) \xrightarrow{\sim} H_i(A)$ for all i where the map on homology takes a chain in X to the same chain in $X \times \{0\} = A$. Now recall from the discussion preceding the problem that M_f deformation retracts onto Y in a way that provides a homotopy from the inclusion $X \hookrightarrow M_f$ to the map $f : X \to M_f$ given by enlarging the codomain. In particular, replacing $H_i(A)$ with $H_i(X)$ and $H_i(M_f)$ with $H_i(Y)$ changes ι_* to f_*, and we obtain the sequence that we want. ∎

The hardest part of making the exact sequence into what we wanted was making sure the map ι_* became the map f_*. This worked because the maps we used to swap out the objects in the sequence actually changed ι into f. In general we can always swap out abstractly isomorphic objects in an exact sequence to get another exact sequence, but to study the maps in the sequence you need to keep track of what the isomorphisms are actually doing.

Try using this exact sequence on the following example.

Fall 2022, #9 The space $S^1 \times S^1$ is the mapping cone of the map

$$[a, b] : S^1 \to S^1 \vee S^1,$$

representing the commutator of the inclusion of the left summand $a : S^1 \to S^1 \vee S^1$ and the inclusion of the right summand $b : S^1 \to S^1 \vee S^1$. Use this and the long exact sequence to compute the homology.

Another construction is the *mapping torus*, which also has an associated exact sequence (although it is more difficult to derive).

Definition 10.4. *If $f : X \to X$ is a map, the* mapping torus T_f *of f is the quotient* $(X \times [0,1])/((x, 0) \sim (f(x), 1))$.

The exact sequence associated to this construction is as follows.

Lemma 10.1. *For T_f the mapping torus of a map $f : X \to X$, there is an exact sequence*

$$\cdots \to H_i(X) \xrightarrow{1-f_*} H_i(X) \xrightarrow{\iota_*} H_i(T_f) \to H_{i-1}(X) \to \cdots,$$

where ι is an inclusion $X \hookrightarrow T_f$.

Proof. We begin with the sequence of pairs for $X \times \{0, 1\} \subset X \times [0, 1]$. We are interested in the portion of the sequence reading

$$\cdots \to H_{i+1}(X \times [0,1],\, X \times \{0,1\}) \xrightarrow{\delta} H_i(X \times \{0,1\}) \xrightarrow{\iota_*} H_i(X \times [0,1]) \to \cdots.$$

We have a quotient map of pairs $q : (X \times [0,1],\, X \times \{0,1\}) \to (T_f, X)$ inducing a map of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r, "0"] & H_{i+1}(X \times [0,1],\, X \times \{0,1\}) \arrow[r, "\delta"] \arrow[d, "q_*"] & H_i(X \times \{0,1\}) \arrow[r, "\iota_*"] \arrow[d, "q_*"] & H_i(X \times [0,1]) \arrow[r] \arrow[d, "q_*"] & \cdots \\ \cdots \arrow[r] & H_{i+1}(T_f, X) \arrow[r, "\delta"] & H_i(X) \arrow[r, "\iota_*"] & H_i(T_f) \arrow[r] & \cdots \end{tikzcd}$$

Analyze the sequence as follows: since $X \times [0,1]$ deformation retracts onto $X \times \{0\}$ and $X \times \{1\}$, the map ι_* is surjective. Thus the following map is the zero map and δ is injective, identifying $H_{i+1}(X \times [0,1], X \times \{0,1\})$ with the kernel of ι_*. But we know $\ker \iota_*$ explicitly: it consists of $(\alpha, -\alpha)$ for $\alpha \in H_i(X)$. So $\ker \iota_* \cong H_i(X)$. Finally, since the pairs involved are good pairs, the map q_* induces an isomorphism on the relative homology groups, so we can replace $H_{i+1}(T_f, X)$ by $H_n(X)$ in the sequence. Finally, passing ι_* through q shows that the corresponding map in the new sequence is $1 - f_*$, as desired. $\square$

This sequence is useful for the following problem.

Fall 2016, #10 If $f : X \to X$ is a self-map, then the mapping torus of f is the quotient

$$T_f := (X \times [0,1]) / ((x,0) \sim (f(x),1)).$$

For $n \in \mathbb{Z}$, let f_n be a degree n map $S^3 \to S^3$. Compute the homology groups of T_{f_n}.

11 Cup product

11.1 Ring structure on Cohomology

Cup product is an operation on cohomology similar to wedge product on deRham cohomology (In particular, it has all the same properties!). Sometimes it allows to distinguish between different spaces.

Definition 11.1. *The cup product of two cochains $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ (viewed as duals of the chain groups) is $\phi \smile \psi \in C^{k+l}(X;R)$ defined on the simplex $\sigma : \Delta^{k+l} \to R$ by*

$$(\phi \smile \sigma)(\sigma) := \phi(\sigma|[v_0, \ldots, v_k])\phi(\sigma|[v_k, \ldots, v_{l+k}].$$

Lemma 11.1.

$$\delta(\phi \smile \phi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi,$$

where $\phi \in C^k(X;R)$.

This allows us to consider the induced cup product $H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R)$.

Definition 11.2. *Define $H^*(X;R) = \bigoplus_n H^n(X;R)$ with the product*

$$(\sum \alpha_i)(\sum \beta_j) = \sum \alpha_i \beta_j,$$

where $\alpha_k, \beta_k \in H^k(X;R)$.

If R has an identity 1, define $1 \in H^0(X;R)$ as the element whose value on each 0-simplex is 1.

This will make $H^*(X;R)$ into a ring.

Consider a couple of properties:

Proposition 11.1. *Suppose R is commutative. Then*

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha,$$

where $\alpha \in H^k(X;R), \beta \in H^l(X;R)$.

Proposition 11.2. *Suppose $f : X \to Y$ has the induced maps $f^* : H^n(Y; R) \to H^n(X; R)$. Then*

$$f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta).$$

That is, f^* induces a ring isomorphism.

Everything above can also be extended to the relative case.

11.2 Künneth Formula

We describe a way to calculate cohomology of a space by decomposing it into smaller spaces whose cohomology rings we already know.

$$H^*(X; R) \otimes H^*(Y; R) \to H^*(X \times Y; R)$$

This map becomes a ring homomorphism if we define the multiplication multiplcation in the tensor product as $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ (with $|x|$ being the dimension of x). Then the map above sends this element to

$$\begin{aligned}(-1)^{|b||c|} ac \times bd &= (-1)^{|b||c|} p_1^*(a \smile c) \smile p_2^*(b \smile d) \\ &= (-1)^{|b||c|} p_1^*(a) \smile p_1^*(c) \smile p_2^*(b) \smile p_2^*(d) \\ &= p_1^*(a) \smile p_2^*(b) \smile p_1^*(c) \smile p_2^*(d) \\ &= (a \times b)(c \times d)\end{aligned}$$

which is the product of the images of $(a \otimes b)$ and $(c \otimes d)$.

The Künneth formula below tells us that this map is an isomorphism.

Proposition 11.3. *(Künneth formula) The cross product $H^*(X; R) \otimes_R H^*(Y; R) \to H^*(X \times Y; R)$ is an isomorphism of rings if X and Y are CW complexes and $H^k(Y; R)$ is a finitely generated free R-module for all k.*

11.3 Computations and examples

Theorem 11.2 (Spaces with polynomial cohomology)**.**

- $H^*(S^n) \equiv \mathbb{Z}[\alpha]/(\alpha^2)$, $|\alpha| = n$
- $H^*(\mathbb{R}P^n, \mathbb{Z}_2) \equiv \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$, $|\alpha| = 1$ *(i.e. $\alpha \in H^1(\mathbb{R}P^n, \mathbb{Z}_2)$.*
- $H^*(\mathbb{R}P^\infty, \mathbb{Z}_2) \equiv \mathbb{Z}_2[\alpha]$, $|\alpha| = 1$
- $H^*(\mathbb{C}P^n, \mathbb{Z}) \equiv \mathbb{Z}[\alpha]/(\alpha^{n+1})$, $|\alpha| = 2$

11.4 Problems

Fall 2015, 7 & Spring 2020, 6 Show that any map from $S^2 \times S^2 \to \mathbb{C}P^2$ is of even degree.

Notice that the ring structure on $H^*(\mathbb{C}P^2)$ is $\mathbb{Z}[\alpha]/(\alpha^3)$, where $\alpha \in H^2(\mathbb{C}P^2)$. Therefore α^2 is the generator of $H^4(\mathbb{C}P^2)$. Since f^* is a ring homomorphism, $f^*(\alpha^2) = (f^*\alpha)^2$.

Notice that $H^2(S^2 \times S^2) \cong H^2(S^2) \otimes H^2(S^2)$ by the Künneth formula. Therefore $f^*(\alpha)$ should be of the form $f^*(\alpha) = c_1\pi_1^*(\theta_1) + c_2\pi_2^*(\theta_2)$, where θ_1 and θ_2 are the volume forms on S^2. But then we get

$$f^*(\alpha^2) = (f^*(\alpha))^2 = c_1^2\pi_1^*(\theta_1^2) + 2c_1c_2\pi_1^*(\theta_1)\pi_2^*(\theta_2) + c_2^2\pi_2^*(\theta_2^2) = 2c_1c_2\pi_1^*(\theta_1)\pi_2^*(\theta_2).$$

As $c_1, c_2 \in \mathbb{Z}$, our map f is of even degree. ■

Spring 2018, 5 A symplectic form on an eight dimensional manifold is a closed 2-form ω such that ω^4 is a volume form. Determine which of the following admits a symplectic form: $S^8, S^2 \times S^6, S^2 \times S^2 \times S^2 \times S^2$.

12 Degree

Like all great topics on this qual, degree can be defined in several different equivalent ways. Different definitions are advantageous for solving different problems, and there has been at least one qual question explicitly asking you to relate the definitions, so it is important that you can juggle them fluently.

12.1 Algebraic Topology

The algebraic topology definition of degree is the narrowest, focusing on maps from $S^n \to S^n$.

Definition: Given a map $f : S^n \to S^n$, the pushforward $f_* : \tilde{H}_n(S^n) \to \tilde{H}_n(S^n)$ is an group homomorphism $\mathbb{Z} \to \mathbb{Z}$ given by multiplying by some $d \in \mathbb{Z}$. This is the *degree of* f, $\deg(f) = d$.

Proposition 12.1. ***Basic Properties:***

(a) $\deg(\mathbb{1}) = 1$.

(b) $\deg f = 0$ *if f is not surjective. (Given $x \notin im(f)$, we can factor $f : S^n \to S^n$ through $S^n \backslash \{x\}$ which is contractible)*

(c) If $f \simeq g$, then $\deg(f) = \deg(g)$. *(Since then $f_* = g_*$.) Moreover, the converse is also true and is known as the Hopf Degree Theorem.*

(d) $\deg(fg) = \deg(f) \cdot \deg(g)$.

(e) If f is a reflection across a hyperplane, $\deg(f) = -1$. *This means* $\deg(-\mathbb{1}) = (-1)^{n+1}$, *and that if f has no fixed points, then* $\deg(f) = (-1)^{n+1}$.

Computing the degree of a general map is difficult, unless there is a clear image of the generator $\Delta_1^n - \Delta_2^n$ for $H_n(S^n)$, like in the reflection map. But we can use the local degree, and this is often easier to work through.

Suppose that we have $f : S^n \to S^n$ and $y \in S^n$ with $f^{-1}(y) = \{x_1, \ldots, x_n\}$ a finite set. Then for each x_i, we can choose an open neighborhood U_i, such that all of the U_i are pairwise disjoint. And we can choose a neighborhood V of y such that each U_i is mapped into V. I.e., $f(U_i \backslash \{x_i\}) \subset V \backslash \{y\}$. (We can tighten the hypotheses a bit so that the inclusion is an equality by appropriately intersecting our sets, but in general this does not matter.) Then for any i, we have a commutative diagram:

$$\begin{tikzcd}
 & H_n(U_i, U_i\backslash\{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, hook] & H_n(V, V\backslash\{y\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n\backslash\{x_i\}) & H_n(S^n, S^n\backslash f^{-1}(y)) \arrow[l, hook'] \arrow[r, "f_*"'] & H_n(S^n, S^n\backslash\{y\}) \\
 & H_n(S^n) \arrow[r, "f_*"'] \arrow[u] \arrow[ul, "\cong"] & H_n(S^n) \arrow[u, "\cong"']
\end{tikzcd}$$

It's important to understand what these maps are, so let's go through them. First, we have two maps induced by the inclusions $(S^n, S^n\backslash f^{-1}(y)) \hookrightarrow (S^n, S^n\backslash\{x_i\})$ and $(U_i, U_i\backslash\{x_i\}) \hookrightarrow (S^n, S^n\backslash f^{-1}(y))$.

Next, from the long exact sequence of the pair $(S^n, S^n\backslash\{p\})$ for any point $p \in S^n$, we get

$$\cdots \to H_n(S^n\backslash\{p\}) \to H_n(S^n) \xrightarrow{\cong} H_n(S^n, S^n\backslash\{p\}) \to H_{n-1}(S^n\backslash\{p\}) \to \cdots$$

where the middle map is an isomorphism because $S^n\backslash\{p\}$ is contractible. And finally, by using excision with $X = S^n, A = S^n\backslash\{x_i\}$ (which is open) and $Z = S^n\backslash U_i$ (which is closed, so $\text{int}(A) \supset \text{cl}(Z)$), we have

$$H_n(U_i, U_i\backslash\{x_i\}) \xrightarrow{\cong} H_n(S^n, S^n\backslash\{x_i\})$$

The upshot is that we have isomorphisms $H_n(U_i, U_i\backslash\{x_i\}) \cong H_n(S^n) \cong \mathbb{Z}$ and $H_n(V, V\backslash\{y\}) \cong H_n(S^n) \cong \mathbb{Z}$, so we can define the *local degree of f at x_i*, written $\deg(f|x_i)$, as the degree of $f_* : H_n(U_i, U_i\backslash\{x_i\}) \to H_n(V, V\backslash\{y\})$. Moreover, using the diagram we have the following proposition:

Proposition 12.2. *Given $f : S^n \to S^n$ and $y \in S^n$ with $f^{-1}(y) = \{x_1, \ldots, x_n\}$ finite, then*

$$\deg(f) = \sum_i \deg(f|x_i)$$

Proof. The main idea of the proof is to use the commutative diagram to write $H_n(S^n, S^n\backslash f^{-1}(y))$ as the direct sum $\bigoplus_i H_n(U_i, U_i\backslash\{x_i\})$ with the given inclusion maps. Then after some diagram chasing the degrees add. □

Exercise: Compute the degree of $f_k : S^1 \to S^1$ given by $f_k : z \mapsto z^k$.

12.1.1 Maps of Arbitrary Degree

Another thing you should be familiar with is with constructing maps $f : S^n \to S^n$ of arbitrary degree $k \in \mathbb{Z}$ for all $n \geqslant 1$. There are two ways of doing this, first with wedge sums (which generalizes) and second with suspensions (which I think is cleaner)

First, take k disjoint open n-disks $D_1^n, \ldots D_k^n \subset S^n$, and consider the quotient

$$\pi : S^n \to S^n/(S^n\backslash \bigcup_i D_i^n) \simeq \bigvee_i S^n$$

Next, for each S^n in $\vee_i S^n$, take the map $f_i : S^n \to S^n$ of degree ± 1, depending on if k is positive or negative. From the local degree, taking the composition of each f_i with π gives a map of $\sum_i \pm 1 = k$.

Another way to do this is with suspensions. First, define the cone $C(S^n) = S^n \times I/(S^n \times \{1\})$. Notably, this has base $S_0^n = S^n \times \{0\}$ and is contractible because it deformation retracts onto the point at the top. Also, $S(S^n) = CS^n/S^n_0$. This means that from the long exact sequence of reduce homology for the pair (CS^n, S^n_0), we have

$$\cdots \to \tilde{H}_{n+1}(CS^n) \to \tilde{H}_{n+1}(CS^n, S^n_0) \xrightarrow{\cong} \tilde{H}_n(S^n_0) \to H_n(CS^n) \to \cdots$$

where we get an isomorphism because CS^n is contractible. This is just the isomorphism $\tilde{H}_{n+1}(S(S^n)) \cong \tilde{H}_n(S^n)$, but we know that the induced map $Sf_* : S^{n+1} \to S^{n+1}$ gives a commuting diagram

$$\begin{tikzcd}
\tilde{H}_{n+1}(S^{n+1}) \arrow[r, "\cong"] \arrow[d, "Sf_*"] & \tilde{H}_n(S^n) \arrow[d, "f_*"] \\
\tilde{H}_{n+1}(S^{n+1}) \arrow[r, "\cong"] & \tilde{H}_n(S^n)
\end{tikzcd}$$

Thus $\deg Sf = \deg f$, so we can construct a map of arbitrary degree on S^1 and then suspend to the desired dimension.

12.2 Intersection Theory

We can also define the degree of a map more generally by using intersection theory. Similar to the mod 2 case, we suppose $f : X \to Y$ is transversal to $Z \subset Y$. Then $f^{-1}(Z)$ is a finite number of points, each with an orientation number -1 or $+1$ depending on the preimage orientation. The intersection number $I(f, Z)$ is the sum of the orientation numbers.

The orientation number at a point $x \in f^{-1}(Z)$ is defined as follows. If $f(x) = z$ then transversality gives

$$df_x Tx(X) \oplus T_z(Z) = T_z(Y)$$

Since df_x must bean isomorphism onto its image, the orientation of X provides an orientation of $df_x T_x(X)$. Then the orientation number at x is $+1$ if the orientations on $df_x T_x(X)$ and $T_z(Z)$ "add up" to the prescribed orientation on Y (in the same order), and -1 otherwise.

Definition: If M^n, N^n are oriented n manifolds with a map $F : M \to N$, and either M is closed and N is connected, or just if F is proper, then we can define

$$\deg(F) = I(F, \{q\}) \quad q \in N$$

To see that this makes sense, consider a regular value $y \in N$ with $F \pitchfork \{y\}$. This means there is some connected neighorhood V of y that is evenly covered by $F^{-1}(V) = \bigcup_i U_i$, and each $F|_{U_i} : U_i \to V$ a diffeomorphism. Thus

$$\deg(F) = I(F, \{y\}) = \sum_i \operatorname{sgn} \det DF|_{x_i}$$

Note that the sign of the determinant is continuous, so this definition is locally constant. Therefore, it is constant on a connected manifold N provided that some regular value exists, and this is guaranteed via Sard's Theorem.

Exercise: Think about how this related to the local degree.

Theorem 12.1. *Even dimensional spheres do not admit non-vanishing vector fields.*

Proof. Viewing the tangent space T_pS^n as $\mathbb{R}^n$, we see that a tangent vector to $p \in S^n$ is a perpendicular vector. Then by normalizing, we can produce a homotopy between $\mathbb{1}$ and $-\mathbb{1}$, which only works if $\deg(-\mathbb{1}) = (-1)^{n+1} = \deg(\mathbb{1})$. □

Theorem 12.2. *If $F : M \to N$ is a proper, nonsingular map with $\deg(F) = \pm 1$ between orientable, connected manifolds, then F is a diffeomorphism.*

Proof. Because $F : M \to N$ is a proper nonsingular map, it is a covering map. Thus because each neighborhood is evenly covered,

$$|\deg(F)| = \#F^{-1}(y) = 1$$

so we see F is bijective. Therefore, F is a diffeomorphism. □

Theorem 12.3 (Hopf Degree Theorem)**.** *Let M^n be a connected, closed, orientable manifold. Then $F_0, F_1 : M \to S^n$ are homotopic only if $\deg(F_0) = \deg(F_1)$.*

For S^n degrees, this is the converse of the statement that homotopic maps induce the same map on homology, so they have the same degree. The proof is involved and I don't think it is not tested on the qual; the result may come up occasionally however.

12.3 Integration

If M^n, N^n are connected, orientable manfiolds and $F : M \to N$ is proper, then we have a commutative diagram

$$\begin{array}{ccc} H_c^n(N) & \xrightarrow[F^*]{} & H_c^n(M) \\ \downarrow \int & & \downarrow \int \\ \mathbb{R} & \xrightarrow[d]{} & \mathbb{R} \end{array}$$

The homomorphism $\mathbb{R} \to \mathbb{R}$ is given by multiplying by some real number d, and we define $\deg(F) = d$. That is, $d \in \mathbb{R}$ such that

$$\int_M F^*\omega = \deg(F) \int_N \omega$$

for some orientation form $\omega \in \Omega_c^n(N)$.

Lemma 12.4. *If $F : M \to N$ is a diffeomorphism, then $\deg(F) = \pm 1$.*

Proof. F can either preserve or reverse orientation, so the sign of the integral is either changed or remains the same. □

12.4 Proof of Equivalence

Given $F : M \to N$ a proper map of connected, orientable n-manifolds, suppose

$$\int_M F^*\omega = \deg(F) \int_N \omega$$

for some orientation form $\omega \in \Omega_c^n(N)$. By Sard's Theorem, there is a regular value $y \in N$. This means there is a connected neighborhood (V, y) which is evenly covered by the neighorhoods (U_i, x_i) in M, where the U_i are mutually disjoint and map diffeomorphically to V.

Now select a form $\omega \in \Omega_c^n(V)$ which integrates to

$$\int_V \omega = 1$$

Note that we can write the pullback $F^*\omega \in \Omega_c^n(M)$ as

$$F^*\omega = \sum_i F^*\omega|_{U_i}$$

and we have

$$\int_{U_i} F^*\omega|_{U_i} = \pm 1$$

because $F|_{U_i} : U_i \to V$ is a diffeomorphism. And the sign depends on if DF_{x_i} preserves or reverses orientation at $x_i \in U_i$, so we see

$$\int_M F^*\omega = \sum_i \int_{U_i} F^*\omega|_{U_i} = \deg(F) \int_M \omega = \deg(F)$$

Thus this degree is exactly the intersection number $I(F, \{y\})$, so the intersection theory definition is the same as the integration theory.

12.5 Problems

12.5.1 Basic Definitions and Constructions

Fall 2012, #4

(a) Show that for any $n \geqslant 1$ and $k \in \mathbb{Z}$, there exists a continuous map $f : S^n \to S^n$ of degree k.

(b) Let X be a compact, oriented n-dimensional manifold. Show that for any $k \in \mathbb{Z}$, there exists a continuous map $f : X \to S^n$ of degree k.

(a) For any $k \neq 0$, choose $|k|$ disjoint open sets $U_1, \cdots, U_{|k|}$, each diffeomorphic to B^k (a k-cell). Consider the map $\pi : S^n \to S^n / \big(S^n - \big(U_1 \cup \cdots \cup U_{|k|}\big)\big) \cong \bigvee_{i=1}^k S^n$. Note that π has degree 1 since it maps S^n to $\bigvee_{i=1}^k S^n$ maintains the same orientation for any open neighborhood of any $x \in S^n$. We now construct $\tilde{f} : \bigvee_{i=1}^k S^n \to S^n$ such that if $k > 0$, each S^n is mapped to S^n via the identity, and if $k < 0$, we map each S^n is mapped to S^n by swapping one of the signs. Since the degree of a composition is the product of the degrees, we note that $\tilde{f} \circ \pi$ has degree k, as desired.

For $k = 0$, we just take any constant map.

(b) Since X is deformation retracts to S^n, we do the same thing as above: take $|k|$ disjoint open sets, define the map $\pi : X/(X - (U_1 \cup U_2 \cdots \cup \ \ U_{|k|})) \cong \bigvee_{i=1}^{k} S^n$ which has degree 1 again, and then take the map $\tilde{f}$ described in part (a)

. ■

Spring 2013, #7 Let $F : S^n \to S^n$ be a continuous map.

(a) Define the degree $\deg(F)$ of F and show that when F is smooth

$$\deg(F) \int_{S^n} \omega = \int_{S^n} F^* \omega$$

for all $\omega \in \Omega^n(S^n)$.

(b) Show that if F has no fixed points then $\deg(F) = (-1)^{n+1}$.

(a) The degree of F is the number $\deg(F)$ such that given a generator $[\alpha]$ of $H_n(S^n)$, we have $F_*[\alpha] = \deg(F)[\alpha]$. In other words, F_* is the multiplication by $\deg(F)$ map on top homology. So, we note that if we view S^n as a n-cycle, then we have

$$\int_{S^n} F^* \omega = \int_{F_* S^n} \omega = \deg(F) \int_{S^n} \omega$$

as desired (since $F_* S^n$ is a $\deg(F)$-fold cover of S^n).

(b) Note that, via Lefschetz theory, if F has no fixed points, then $L(F) = 0$. However,

$$\begin{aligned} 0 = L(F) &= \sum_{i=0}^{n} (-1)^i \operatorname{tr}(F_* : H_i(S^n) \to H_i(S^n)) \\ &= \operatorname{tr}(F_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \operatorname{tr}(F_* : H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \deg(F). \end{aligned}$$

■

Spring 2014, #3 Let S^n be the unit sphere. Determine the values of $n \geqslant 0$ for which the antipodal map $S^n \to S^n, x \mapsto -x$ is isotopic to the identity.

An isotopy is homotopy connecting two given homeomorphisms. We first consider the case n is odd, where in this case we will explicitly construct such a homotopy $H(v,t)$ for $v \in S^n, t \in [0,1]$. Since $n+1$ is even, consider $H(v,t) = M_t v$, where M_t is the $(n+1) \times (n+1)$ block diagonal matrices with block diagonal entries

$$\begin{bmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{bmatrix}$$

.

and zeroes everywhere else. We note that for every t, M_t is a diffeomorphism, and that moreover, $M_0 = I$ (the identity) and $M_1 = -I$ (the antipodal map). Thus, for n odd, we have the antipodal and identity being isotopic.

For the case when n is even, we will use the invariance of degree under homotopy to show that these two maps are not homotopic. Recall that the antipodal map has degree $(-1)^{n+1}$ which equals -1 when n is even. Suppose they now that they are isotopic. Let id be the identity and A be the antipodal map. In this case, we have $id^* = A^*$, which means that given any n-form ω, we have,

$$\int_{S^n} \omega = \int_{S^n} id^*(\omega) = \int_{S^n} A^*(\omega) = \deg(A) \int_{S^n} \omega = -\int_{S^n} \omega$$

so $\int_{S^n} \omega = 0$ for any ω. In particular, if we let $\omega = x_1 dx_2 \wedge \cdots \wedge dx_n$, we have by Stokes that

$$\int_{S^n} \omega = \int_{B^{n+1}} d\omega = \int_{B^{n+1}} dx_1 \wedge \cdots \wedge dx_n > 0$$

Thus, in this case, the identity and antipodal maps are not isotopic. ∎

12.5.2 Cohomology Ring

Theorem 12.5 (Universal Coefficient Theorem for Cohomology)**.** *If a chain complex C of free abelian groups has homology groups $H_n(C)$. Then the cohomology groups $H^n(C;G)$ of the cochain complex* $\mathrm{Hom}(C_n, G)$ *are determined by the split exact sequences*

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \to 0$$

This implies that if M and N are closed, connected, oriented n-manifolds and $f : M \to N$ is a continuous map, then there is a commutative diagram

$$\begin{array}{ccc} H^n(N) & \xrightarrow{\simeq} & \mathrm{Hom}(H_n(N), \mathbb{Z}) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \mathrm{Hom}(f_*, \mathbb{Z})} \\ H^n(M) & \xrightarrow{\simeq} & \mathrm{Hom}(H_n(N), \mathbb{Z}) \end{array}$$

whre the horizontal rows are isomorphisms since $H_{n-1}(M)$ and $H_{n-1}(N)$ are free, so

$$Ext^1(H_{n-1}(M), \mathbb{Z}) = Ext^1(H_{n-1}(N), \mathbb{Z}) = 0$$

Therefore if f_* on top Homology is multiplication by some integer, then f^* on top Cohomology is multiplication by the same integer. More formally, $f^* : H^n(N) \to H^n(M)$ will map a generator of $H^n(N)$ to $\deg(f)$ times a generator of $H^n(M)$.

Spring 2016, #5 Let M be a compact oriented n-manifold with de Rham cohomology group $H^1_{dR}(M;\mathbb{R}) = 0$ and let T^n be the n-dimensional torus. For which integers k does there exist a smooth map $f : M \to T^n$ of degree k?

Spring 2018, #8 Determine all of the possible degrees of maps $S^2 \to S^1 \times S^1$.

12.5.3 General Problems

Fall 2013, #3 Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds of dimensions m and n, respectively, such that $m + n = p$. Suppose that $M \cap N = \varnothing$. Consider the linking map

$$\lambda : M \times N \to S^p, \quad \lambda(x, y) = \frac{x - y}{|x - y|}$$

The degree of λ is called the linking number $\ell(M, N)$.

(a) Show that $\ell(M, N) = (-1)^{(m+1)(n+1)}\ell(N, M)$

(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ disjoint from N, then $\ell(M, N) = 0$.

Fall 2017, #7 Let M be a smooth, compact, connected, oriented n-dimensional manifold (without boundary).

(i) Show that if the Euler characteristic of M is zero, then M admits a nowhere vanishing vector field.

(ii) If M is a surface of genus g, then what is the $\min_v$(#zeros of v), where v ranges over vector fields on M whose zeros are isolated and have index ± 1? Give a proof.

13 Lie Groups

The purpose of this section is mainly to write out one key result – the parallelizability of Lie groups. We'll also add some discussion of how to recognize Lie groups and about when submanifolds of a Lie group is a Lie group. See Peterson's notes for more details, especially regarding polar decomposition.

A Lie group is a smooth manifold with group structure. Essentially, we have everything a smooth manifold has, in addition to the following smooth map: $P : G \times G \to G$ where $(g, h) \mapsto gh$.

We also have the map $L_g : G \to G$ where $h \mapsto gh$, the left multiplication map (also called left translation). This map is a diffeomorphism (inverse is $L_{g^{-1}}$), and is perhaps the most useful property of Lie groups (for the Geometry qual at least).

The main reason for the creation of this section is the following result, which has appeared in some quals recently.

Theorem 13.1 (Parallelizability of Lie groups, Fall 2017 #2, Fall 2022 #2)**.** *Lie groups are parallelizable.*

Proof. We show that G admits a global frame.

Consider the vectors $v_1, \dots, v_n$ which form a basis for T_eG. We define that $w_1, \dots, w_n$ which form a basis for T_gG.

Consider a vector $v_e \in T_eG$, and define the vector $v_g = dL_g(v_e)$. Define the vector field X such that $X(g) = v_g$.

To see X is smooth, pick $\gamma : (-\epsilon, \epsilon) \to G$ such that $\gamma(0) = e$ and $\gamma'(0) = v_i$. Then consider the map $\phi : G \times (-\epsilon, \epsilon) \to G$ such that

$$(g, t) \mapsto P(g, \gamma(t)) = L_g \circ \gamma(t).$$

Then, we note that

$$v_g = \frac{\partial}{\partial t}\bigg|_{t=0} \phi(g,t).$$

Letting $(v_e)_1, \ldots, (v_e)_n$ be a basis for T_eG, as L_g is a diffeomorphism, we note that dL_g is an isomorphism, so $(v_g)_1, \ldots, (v_g)_n$ is a basis for T_gG. Thus, $X_1, \ldots, X_n$ is a global frame for G. $\square$

Note that this result implies all Lie groups are orientable. We also note that all of these vector fields are left-invariant, which is important for Euler characteristic considerations.

There is one more result that is useful: Lie groups always admit a nowhere vanishing vector field. We will go into the construction of this vector field, however it does not immediately imply that all Lie groups have trivial Euler characteristic via Poincare-Hopf. In order to apply this result, we need our Lie group G to be homotopically equivalent to a compact Lie group. For example, $SL_n(\mathbb{C})$ is homotopically equivalent to $SO_n(\mathbb{C})$ (one way to show this is via a deformation retraction using polar decomposition).

Theorem 13.2. *Lie groups admit a nowhere vanishing (left-invariant) vector field.*

Proof. Define the vector field $X : G \to TG$ where

$$X(h) = dL_h(X(e)).$$

This is left-invariant as

$$dL_g(X(h)) = dL_g(dL_h(X(e)) = dL_{gh}(X(e)) = X(gh).$$

$\square$

14 Flows and Lie Derivatives

Most of this content is from Lee Chapter 9, though with far less detail.

14.1 Flows

14.1.1 Integral Curve Precursors

Definition 14.1 (Integral Curve)**.** *If M is a manifold (with or without boundary) and V is a vector field on M, an* integral curve *of V is a smooth curve $\gamma : J \to M$ with*

$$\gamma'(t) = V_{\gamma(t)}.$$

Remember that $\gamma'(t) = \gamma_*(\frac{d}{dt})$. We say that $\gamma(0)$ is the *starting point* of γ.

As a baby version of the Fundamental Theorem on Flows (which we'll get to), one can cook up a curve integral to V from any starting point $p \in M$.

Proposition 14.1. *V is a smooth vector field on M. For each $p \in M$, there is a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ that is an integral curve of V starting at p.*

In many problems, it is more convenient to cook up an integral curve in an open subset of $\mathbb{R}^n$ and push it forward to an integral curve on M by a chart.

Proposition 14.2. *Let $F : M \to N$ be a smooth map and X, Y vector fields on M and N respectively. Then X and Y are F-related iff for each integral curve γ of X, $F \circ \gamma$ is an integral curve of Y. Recall that X and Y are F-related if $F_* X = Y$.*

14.1.2 Flows

Definition 14.2. *A* global flow *is a continuous left $\mathbb{R}$-action on M; that is, a continuous map $\theta : \mathbb{R} \times M \to M$ satisfying:*

- *For each t define $\theta_t : M \to M$ by $\theta_t(p) = \theta(t, p)$. Then*

$$\theta_t \circ \theta_s = \theta_{t+s},\ \theta_0 = \mathbb{1}_M.$$

  The maps θ_t are diffeomorphisms.

- *For each $p \in M$, define a curve $\theta^{(p)} : \mathbb{R} \to M$ by $\theta^{(p)}(t) = \theta(t, p)$. The image of this curve is the orbit of p under the group action.*

To each global flow $\theta : \mathbb{R} \times M \to M$ there is an associated smooth vector field $V_p = \theta^{(p)\prime}(0)$ called the *infinitesimal generator of* θ. V "generates" θ in the sense that each curve $\theta^{(p)}$ is an integral curve of V.

The utility of flows comes from being able to go backwards; that is start from a vector field V on M and find a flow θ that has V as its infinitesimal generator. There are insurmountable obstructions that keep us from always being able to find a *global* flow, since one can cook up a vector field on $(0, 1)$, say, which has an integral curve that might blow up. Check out Lee example 9.9 and 9.10. I could see a qual question asking us to provide such an example so this might be good to check out.

Example 14.1 (Integral curve that can't be extended)**.** *If $M = \mathbb{R}^2$ and $W = x^2 \frac{\partial}{\partial x}$, then one can check that the integral curve of W starting at $(1, 0)$ is*

$$\gamma(t) = ((1 - t)^{-1}, 0),$$

which cannot be extended past $t = 1$.

But one can do the next best thing and define a maximal flow (one that cannot be extended to a larger domain like $\mathcal{D} \subset \mathbb{R} \times M$ with $\mathcal{D}^{(p)} = \{t : (t, p) \in \mathcal{D}\}$ containing 0). You can adjust the definition of maximal flows to get the one for global flows by replacing $\mathbb{R} \times M$ with $\mathcal{D}$.

Theorem 14.1 (Fundamental Theorem on Flows)**.** *Let V be a vector field on M (WITHOUT BOUNDARY). There is a maximal flow $\theta : \mathcal{D} \to M$ with infinitesimal generator V. We have the following properties:*

- *$\theta^{(p)}$ is a unique maximal integral curve of V starting at p.*
- *Some others that aren't too important but you can read them in Lee.*

Flows often show up when we want to find a diffeomorphism on a manifold M that sends some collection of points to another collection, or rotates tangent vectors, etc. One might do this by flowing our points of interest along a well-chosen vector field. Though we really want our flow to exist for all time for something like this to work. A vector field that generates a global flow is called *complete*. All we really need to know is this:

Theorem 14.2. *Every compactly supported smooth vector field on a smooth manifold is complete. In particular every smooth vector field on a smooth compact manifold is complete.*

Here are a couple qual questions that involve flows (but not Lie Derivatives).

Spring 2011, 1. If V is a smooth vector field on an n-manifold M and $V_p \neq 0$ for some $p \in M$, show that we may find a chart (U, x) around p with $V = \frac{\partial}{\partial x^1}$.

Comments: This is Theorem 9.22 in Lee. This comes up a lot. It in particular tells you that you can extend a vector field locally to a basis. Although there might be an easier way to establish this (yes there is. $V = V^i \frac{\partial}{\partial x_i}$. Some coordinate is nonzero in a small neighborhood, say V^1. Then $V, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$ is a local frame). Compare this with Theorem 9.46 about *commuting frames*.

Definition 14.3. *A* commuting frame *is a local frame* (E_i) *for* M *such that* $[E_i, E_j] = 0$.

Theorem 14.3 (Lee 9.46)**.** *Let* M *be an* n*-manifold and* $(V_1, \ldots, V_k)$ *a linearly independent* k*-tuple of smooth commuting vector fields on an open* $W \subset M$. *For each* $p \in W$, *there is a smooth coordinate chart* $(U, (x^i))$ *centered at* p *so that* $V_i = \frac{\partial}{\partial x^i}$ *for* $i = 1, \ldots, k$.

Further compare this with Frobenius theorem and involutivity (the closure under Lie derivative condition). 9.46 is just a more involved application of flows.

Fall 2010, 1 Let M be a connected smooth manifold. Show that for any two non-zero tangent vectors $v_1 \in T_{x_1}M$ and $v_2 \in T_{x_2}M$, there is a diffeomorphism $\phi : M \to M$ such that $\phi(x_1) = x_2$ and $d\phi(v_1) = v_2$.

We show the existence of diffeomorphisms $\phi_{x,y}, \phi_{v,w} : M \to M$ such that $\phi_{x,y}(x) = y$, $\phi_{v,w}(z) = z$ and $d\phi_{v,w}(v) = w$ for any choices $x, y \in M$ and $v, w \in T_zM$. Our diffeomorphism ϕ is then $\phi = \phi_{d\phi(v_1),v_2} \circ \phi_{x_1,x_2}$.

We first show that, for $x, y \in M$, the relation $x \sim y$ iff there exists a diffeomorphism $\phi_{x,y} : M \to M$ such that $\phi_{x,y}(x) = y$ is an equivalence relation. It is clearly reflexive (take $\phi_{x,x} = \mathrm{id}$), clearly symmetric (given $\phi_{x,y}$, we get $\phi_{y,x} = \phi_{x,y}^{-1}$), and clearly transitive (given $\phi_{x,y}$ and $\phi_{y,z}$, we get $\phi_{x,z} = \phi_{y,z} \circ \phi_{x,y}$). As M is connected and equivalence classes partition the manifold M, if we can show the equivalence classes are open, we must have only one equivalence class, implying the first result.

Let $x \in M$ be given, and consider a coordinate patch $(U, \psi = (x^1, \ldots, x^n))$ of x such that U is homeomorphic to $B(0,1) \subset \mathbb{R}^n$ and $\psi(x) = 0$. Take $V \subset \overline{V} \subset U$ such that V is homeomorphic $B(0,r) \subset \mathbb{R}^n$ for some $r < 1$. Let $y \in V$ be given and let $c = (c^1, \ldots, c^n) = \psi(y)$. Consider the vector field $X = c^i \frac{\partial}{\partial x^i}$, which is well-defined on U. Letting α be a bump function such that $\alpha \equiv 1$ on V and $\operatorname{supp} \alpha \subset U$, we note that α is compactly supported. Thus, letting $\tilde{X} = \alpha X$, we have a globally defined compactly supported vector field, thus we have a global flow θ_t. Note additionally that $\tilde{X}|_V = X$.

Noting that the integral curve $\gamma(t) = \psi^{-1}(tc)$ is an integral curve of $\tilde{X}$ satisfying $\gamma(0) = x$ and $\gamma(1) = y$, we have $\theta_1(x) = y$. We define $\phi_{x,y} = \theta_1$. Thus the equivalence class containing x is open and by the arguments above, this implies it is all of M.

We now show the second claim, first proving it in $\mathbb{R}^n$ for $n \geqslant 2$. Let $v, w \in \mathbb{R}^n$ be given, and define $\lambda = \frac{||w||}{||v||} > 0$. If v, w are colinear, define $F_t = \lambda^t \mathrm{id}_{\mathbb{R}^n}$. If not, P be the plane spanned by v and w

with an orientation so that $\{v, w\}$ is a direct basis. Let θ be the oriented angle between v and w. If $w = -\lambda v$, instead consider a third vector w' which is not colinear to v and consider the plane P spanned by v and w' and take $\theta = \pi$. For $t \in \mathbb{R}$, define R_t to be linear isomorphism of $\mathbb{R}^n$ such that its restriction to P is the rotation of angle $t\theta$, and its restriction to $P^\perp$ is the identity map. Take $F_t = \lambda^t R_t$.

By construction, F_t is a 1-parameter family of diffeomorphisms, so let X be its infinitesimal generator. Letting $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a bump function such that $\alpha \equiv 1$ on $B(0, \frac{r}{2})$ and $\alpha \equiv 0$ on $B(0, r)^c$, we can define the vector field $\tilde{X}$ via $\tilde{X} = \alpha X$. Consider $\tilde{F}_t$ as its flow, then we note that $\tilde{F}_1$ has the desired properties.

To obtain our desired solution in our manifold M, letting $z \in M$ be given, there exists some chart (V, ψ) such that $V \subset U$ and $\psi : U \to \mathbb{R}^n$ is a diffeomorphism with $\psi(z) = 0$. By above, there exists a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$ and $d_0 f(d_z\psi(v)) = d_z\psi(w)$. Defining $\phi = \psi^{-1} \circ f \circ \psi$, extended by the identity outside of V via a partition of unity yields our desired diffeomorphism. ∎

14.2 Lie Derivatives

For vector fields, it seems that we just need to remember $\mathcal{L}_V W = [V, W]$. I guess it should be mentioned somewhere that

$$\omega([X, Y]) = X(\omega(Y)) - Y(\omega(X)) - d\omega(X, Y).$$

Definition 14.4 (Lie Derivative of form/covariant tensor field). *Let A be a covariant tensor field on M (this might seem overly general, but this sometimes shows up on the qual; see Fall 2013 Problem 6), and V a vector field on M. If θ is the (local) flow generated by V,*

$$(\mathcal{L}_V A)_p = \frac{d}{dt}\Big|_{t=0} (\theta_t^* A)_p = \lim_{t \to 0} \frac{(\theta_t^* A)_p - A_p}{t}.$$

This is defined pointwise in the vector fields.

Proposition 14.3 (Properties of Lie Derivatives). *The Lie derivative on tensor fields is uniquely determined by the following properties*

(a) *$\mathcal{L}$ is linear over $\mathbb{R}$*

(b) *$\mathcal{L}_V f = Vf$ for functions (0-tensors) f*

(c) *$\mathcal{L}_V(A \otimes B) = \mathcal{L}_V A \otimes B + A \otimes \mathcal{L}_V B$,*

(d) *$\mathcal{L}_V(T(X_1, \ldots, X_k)) = (\mathcal{L}_V T)(X_1, \ldots, X_k) + \sum_{i=1}^k T(X_1, \ldots, L_V(X_i), \ldots, X_k)$, T a k-tensor.*

Specializing to differential forms, we have the extremely useful

Theorem 14.4 (Cartan's Magic Formula). *If V is a vector field on M and ω a differential form,*

$$\mathcal{L}_V \omega = i_V(d\omega) + d(i_V \omega),$$

or more succinctly

$$\mathcal{L}_V = i_V d + d i_V.$$

As a consequence Lie derivatives commute with exterior derivatives. However interior multiplication i_V certainly does not commute with exterior derivatives!

*It's possible that Cartan Magic formula works for general covariant tensors too! Exterior derivative still makes sense. Define it for functions and then extend.

It's nice to note that the product rule

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V\omega) \wedge \eta + \omega \wedge (\mathcal{L}_V\eta)$$

has no sign change because there is no change in grading when $\mathcal{L}_V$ is applied.

14.3 Some computation rules

It looks like interior multiplication wasn't introduced earlier, so I might as well include a couple rules to simplify computations arising in Cartan's magic formula.

Interior multiplication is $i_V\omega(V_1, \dots, V_{k-1}) = \omega(V, V_1, \dots, V_{k-1})$.

Proposition 14.4 (Computation rules for interior multiplication)**.**

(a) $i_V^2 = 0$ *(same as d).*

(b) $i_V(\omega \wedge \eta) = (i_V\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge (i_V\eta)$ *(same as d).*

(c) If $\omega_0, \dots, \omega_n$ *are* 1*-forms, then*

$$i_V(\omega_1 \wedge \cdots \wedge \omega_k) = \sum_{i=0}^{k}(-1)^i\omega_i(V)\omega_1 \wedge \cdots \wedge \hat{\omega}_i \wedge \cdots \wedge \omega_k.$$

14.3.1 Problems

Fall 2010, 4

(a) Let $f_0, f_1 : M \to N$ be smooth. Define the notion of a chain homotopy between f_0^* and f_1^*.

(b) Let X be a smooth vector field on compact manifold M. Let $\phi_t : M \to M$ be the flow generated by X. Find an explicit chain homotopy between ϕ_0^* and ϕ_1^*. *Hint:* Recall Cartan's magic formula.

Spring 2011, 2

(a) Show Cartan's magic formula: $\mathcal{L}_X = di_X + i_Xd$.

(b) Use this to show that a vector field X on $\mathbb{R}^3$ has local flows preserving volume if and only if it has divergence 0.

Spring 2018, 2 Let $\Phi_N, \Phi_S : \mathbb{R} \times S^2 \to S^2$ be two global glows on the sphere S^2. Show that there is an $\epsilon > 0$ and a neighborhood U of the north pole, V of the south pole, and global flow $\phi : \mathbb{R} \times S^2 \to S^2$ such that $\Phi(t, q) = \Phi_N(t, q$ for $t \in (-\epsilon, \epsilon)$, $q \in U$ and $\Phi(t, q) = \Phi_S(t, q)$ for all $t \in (-\epsilon, \epsilon)$ and $q \in V$.

15 Practice Qual Solutions

15.1 Fall 2020

Fall 2020, 1 Let $x_1, x_2, \ldots, x_k$ and $y_1, y_2, \ldots, y_k$ be two sets of distinct points in a connected smooth manifold M with $\dim(M) > 1$, and $v_1, v_2, \ldots, v_k$ and $w_1, w_2, \ldots, w_k$ be the corresponding two sets of non-zero tangent vectors at these points. Show that there is a diffeomorphism f of M such that $f(x_i) = y_i$ and $df_{x_i}(v_i) = w_i$ for $i = 1, \ldots, k$.

This problem is *long*. Here is a good resource in case you want to see a different write-up.

We first prove there exists contractible open subsets $U_1, \ldots, U_k$ with $x_i, y_i \in U_i$, such that their closure is contractible and are pairwise disjoint. We prove this by induction.

For $k = 1$: if $x_1 = y_1$, take a small contractible neighbourhood of x_1 for U_1. If $x_1 \neq y_1$, take any small tubular neighbourhood of an injective path joining x_1 and y_1 for U_1.

Assume $x_1, \ldots, x_{k+1}, y_1, \ldots, y_{k+1}$, are as in the statement of the Lemma. By induction hypothesis, let $U_1, \ldots, U_k$ be disjoint contractible open subsets such that $x_i, y_i \in U_i$ for $1 \leqslant i \leqslant k$, and whose closures are pairwise disjoint. Take them small enough such that $x_{k+1}, y_{k+1} \notin U_i$ for any i. If $x_{k+1} = y_{k+1}$, any small contractible neighbourhood of x_{k+1} works for U_{k+1}. Assume then that $x_{k+1} \neq y_{k+1}$. Since $\overline{U}_1, \ldots, \overline{U}_k$ are contractible, $M \backslash \left(\overline{U}_1 \cup \cdots \cup \overline{U}_k\right)$ is homotopy equivalent to $M \backslash \{k \text{ points}\}$. In particular, it is path-connected. Since $\widetilde{M} = M \backslash \left(\overline{U}_1 \cup \cdots \cup \overline{U}_k\right)$ is an open subset of M, it is then a connected manifold of dimension n, with x_{k+1}, y_{k+1}. Apply the $k = 1$ case to $\widetilde{M}$ in order to conclude the proof of our claim.

Next, we prove that there exists First, prove that there exists diffeomorphisms $f_i : M \to M$ compactly supported on U_i (and extended to identity elsewhere) such that $f_i(x_i) = y_i$ and $df_{x_i}(v_i) = w_i$. This is written in detail in our write-up of Fall 2010, 1. We then let $f = f_1 \circ \cdots \circ f_k$. ∎

Fall 2020, 2 Let M be a smooth manifold of dimension n. Let $T^*M := \bigsqcup_{m \in M} T^*_m M$ be the cotangent bundle, where $T^*_m M$ is the dual of the tangent space $T_m M$, and let $\pi : T^*M \to M$ be the natural projection such that $\pi(\phi) = m$ for $\phi \in T^*_m M$. Let $x = (x_1, \ldots, x_n)$ be local coordinates on $U \subset M$. Then we endow $\pi^{-1}(U)$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, such that the element $\phi \in \pi^{-1}(U)$ with $\pi(\phi) = m$ is written as $\sum_i y_i dx_i(m)$.

(a) Show that T^*M is a smooth manifold with respect to the local coordinate charts defined above.

(b) Define the 1-form λ on the cotangent bundle T^*M as follows: for any tangent vector $v \in T_\phi(T^*M)$ at $\phi \in T^*M$, we set $\lambda(\phi)(v) = \phi(d\pi(v))$.
Find the explicit expression of λ with respect to the above local coordinate chart. Use this to show that λ is smooth.

(c) Find the explicit expression of $d\lambda$ and its k-th exterior powers for all $k \geqslant 2$ with respect to the local coordinate chart above. Use this to show that T^*M is orientable.

(a) There are many resources for this, see for one.

(b) To find[2] an explicit expression of λ, we see what λ does to the basis vectors, $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$. However,

[2] λ is known as the tautological 1-form of T^*M. To see a better explanation, see Lee Proposition 22.11.

we see that as $d\pi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}$ and $d\pi(\frac{\partial}{\partial y_i}) = 0$, we have

$$\lambda(\phi)\left(\frac{\partial}{\partial y_i}\right) = \phi\left(d\pi\left(\frac{\partial}{\partial y_i}\right)\right) = \phi(0) = 0$$

and

$$\lambda(\phi)\left(\frac{\partial}{\partial y_i}\right) = \phi\left(d\pi\left(\frac{\partial}{\partial x_i}\right)\right) = \phi\left(\frac{\partial}{\partial x_i}\right) = y_i.$$

Thus,

$$\lambda = \sum_i y_i dx_i.$$

λ then is clearly smooth because its component functions in these coordinates are linear.

(c) Given the above, we note that

$$d\lambda = d\sum_i y_i dx_i = \sum_i dy_i \wedge dx_i.$$

Thus,

$$(d\lambda)^k = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} k!\, dy_{i_1} \wedge dx_{i_1} \wedge \cdots \wedge dy_{i_k} \wedge dx_{i_k}.$$

In particular,

$$(d\lambda)^n = n!\, dy_1 \wedge dx_1 \wedge \cdots \wedge dy_n \wedge dx_n.$$

Thus, $(d\lambda)^n$ is a nowhere vanishing top-form of T^*M, implying T^*M is orientable.

■

Fall 2020, 3 Let M be a smooth manifold with smooth boundary ∂M and N be a smooth manifold without boundary. Assume that $f : M \to N$ is smooth (this includes smoothness at points of ∂M) so that the tangent map $df_x : T_xM \to T_{f(x)}N$ is well-defined (including at points of ∂M). Let $y \in N$ be a regular value for both f and $f|_{\partial M}$.

(a) Show that $M_1 := f^{-1}(y)$, if not empty, is a smooth submanifold with boundary in M such that the boundary $\partial M_1 = (f_{\partial M})^{-1}(y) = M_1 \cap \partial M$ is a submanifold of ∂M.

(b) If we only assume that y is a regular value for f but not for $f_{\partial M}$, does the conclusion of (a) still hold?

(a) If you'd like to see another write-up, see page 61 of Guillemin & Pollack. First, notice that the interior $\mathrm{Int}(M)$ of M is a smooth manifold without boundary, so $M_1 \cap \mathrm{Int}\, M = (f_{\mathrm{Int}(M)})^{-1}(y)$ is also a smooth manifold without boundary. Therefore it is enough to look at M_1 around a point $x \in \partial M$.

First, move everything in local coordinates. By taking local coordinates in N, we can view a neighborhood of y as a ball in $\mathbb{R}^n$, centered at 0 (i.e. we can assume $y = 0$). Similarly, by taking slice charts in M, we can view the neighborhood of x in M as a subset of $H^m = \{x \in \mathbb{R}^n | x_m \geqslant 0\}$ such that $x = 0$. Moreover, since f is smooth at 0, by shrinking the neighborhood, we can assume that f is defined and smooth in a ball B around 0.

Now, let $S = f^{-1}(0) \cap B$. Then by the Regular Value Theorem, S is a smooth manifold of codimension 1. Define also $\pi : S \to \mathbb{R} : x \mapsto x_m$. Notice that $B \cap M_1 = \pi^{-1}[0, +\infty]$. We will now show that 0 is a regular value of π. For that, notice that since 0 is a regular value of both

f and $f|_{\partial M}$, both df_0 and $d(f|_{\partial M})_0$ are surjective, so their kernels have the same codimension n. But that means that these kernels have different dimensions (because the lie in $T_0\partial M$ and $T_0 M$, which have different dimensions). But that means that $\ker df_0 \neq \ker d(f|_{\partial M})_0$. That is $T_0 S = \ker df_0$ does not lie in $T_0 M$ entirely, since $df_0|_{T_0\partial M} = d(f|_{\partial M})_0$. That means that locally there exists a vector field $X \in T_0 S$, $X \notin T_0\partial M$ (or in other words, $X_m \neq 0$). But notice that $d\pi_0(X) = X_m \neq 0$, so $d\pi_0$ is surjective. That means, 0 is a regular value of π.

The proof then follows from the following lemma: Suppose that S is a manifold without boundary and that $\pi : S \to \mathbb{R}$ is a smooth function with regular value 0. Then the subset $\{s \in S \mid \pi(s) \geqslant 0\}$ is a manifold with boundary, and the boundary is $\pi^{-1}(0)$.

This lemma is true as the set $\pi(s) > 0$ is open in S and is therefore a submanifold of the same dimension as S.This is true here as, around 0, π is locally equivalent to the canonical submersion near 0. But the above lemma is clear when π is the canonical submersion.

(b) The answer is no. For a counterexample, let $M = \{y \geqslant 0\} \subset \mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto y$. Then it is easy to see that as f is linear, $df = f$, so f is a submersion. In particular, $y = 0$ is a regular value of f. At the same time $\partial M = \{y = 0\}$, so $f|_{\partial M} = 0$, which means that 0 is not a regular value of $f|_{\partial M}$.

In this notation, $M_1 = f^{-1}(0) = \{y = 0\} = \partial M$. Then M_1 is a manifold without boundary, but $M_1 \cap \partial M = \partial M \neq \varnothing = \partial M_1$. So the conclusion does not hold.

■

Fall 2020, 4 Let S be a closed subset of a smooth manifold M that has a second countable topological basis. Show that for any positive integer n, there is a smooth map $f : M \to \mathbb{R}^n$ such that $S = f^{-1}(0)$.

This is Theorem 2.29 in Lee. ■

Fall 2020, 5 Let $M, N \subset \mathbb{R}^{p+1}$ be two compact, smooth, oriented submanifolds (without boundary) of dimensions m and n, respectively, such that $m + n = p$, and suppose that $M \cap N = \varnothing$. Let $l(M,N)$ be the degree of the map

$$\lambda : M \times N \to S^p, \quad \lambda(x,y) = \frac{x-y}{||x-y||}.$$

(a) Show that $l(M,N) = (-1)^{(m+1)(n+1)} l(N,M)$.
(b) Show that if M is the boundary of an oriented submanifold $W \subset \mathbb{R}^{p+1}$ which is disjoint from N, then $l(M,N) = 0$.

(a) Denote λ_M as the map $\lambda : M \times N \to S^p$. Consider the composition of maps $\lambda_N = A \circ \lambda_M \circ S$, where $S : N \times M \to M \times N$ is the map $s(n,m) = (m,n)$ and A is the antipodal map on S^p. Note that a flip and a reflection are both orientation reversing diffeomorphisms, implying their degree is -1. As A is the composition of $p+1$ reflection, we have $\deg(A) = (-1)^{p+1} = (-1)^{m+n+1}$ and as S is the composition of mn flips, we have $\deg(S) = (-1)^{mn}$. Thus,

$$\begin{aligned} l(N,M) = \deg(\lambda_N) = \deg(A \circ \lambda_N \circ S) &= \deg(A)\deg(\lambda_M)\deg(S) \\ &= (-1)^{m+n+1} l(M,N)(-1)^{mn} = (-1)^{(m+1)(n+1)} l(N,M). \end{aligned}$$

(b) Let F be the map, $F : W \times N \to S^p$ where

$$F(w, n) = \frac{w - n}{||w - n||}.$$

We note that $\lambda = \partial F = F \circ i_M$ where $i_M : M \times N \to W \times N$ is the inclusion map, as $\partial(W \times N) = \partial W \times N = M \times N$ as N has no boundary. Then, letting ω be a volume form in S^p, we have, via Stokes,

$$\begin{aligned} \deg(\lambda) \int_{S^p} \omega = \int_{M \times N} \lambda^* \omega = \int_{M \times N} i_M^* F^* \omega \\ = \int_{W \times N} dF^* \omega = \int_{W \times N} F^*(d\omega) = \int_{W \times N} F^*(0) = \int_{W \times N} 0 = 0, \end{aligned}$$

where $d\omega = 0$ since ω is a top form. Thus $\deg(\lambda) = 0$ as $\int_{S^p} \omega \neq 0$ as ω is a volume form.

■

Fall 2020, 6 Let X be a topological space and let $Y = X \times [-1, 1]/\sim$, where

$$\begin{aligned} (x, -1) \sim (x, -1) \quad \text{for all } x \in X \\ (x, 1) \sim (x, 1) \quad \text{for all } x \in X \end{aligned}$$

Describe the relationship between the homology groups of X and Y.

$Y = S(X)$, the suspension of X. Follow the argument as in 10.1 to get $\tilde{H}_k(Y) = \tilde{H}_{k-1}(X)$ for all k. ■

Fall 2020, 7

(a) Describe a cell decomposition for $X = \mathbb{R}P^4$ such that its 2-skeleton $X^{(2)} = \mathbb{R}P^2$. (This means that X is obtained from $X^{(2)}$ by attaching only 3- and 4-dimensional cells.) Include a careful description of the attaching maps

(b) Use your cell decomposition to compute $H_k(X; \mathbb{Z})$ and $H_k(X, X^{(2)}; \mathbb{Z})$ for all $k \geqslant 0$.

(a) We construct a CW complex for $\mathbb{R}P^n$ which has $X^{(k)} = \mathbb{R}P^k$ for every $0 \leqslant k \leqslant n$. Our CW complex will have exactly 1 k-cell for $0 \leqslant k \leqslant n$, denoted as e^k. Our attaching maps are

$$\phi_k : S^{k-1} \to X^{(k-1)} = \mathbb{R}P^{k-1},$$

the double cover of $\mathbb{R}P^{k-1}$, where $x, -x \mapsto [x]$. To compute the boundary map, d_k, we note that we have the composition

$$S^{k-1} \xrightarrow{\phi_k} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \cong S^{k-1},$$

with q a quotient map. Note that the map $q\phi$ restricts to a homeomorphism from each component of $S^{k-1} - S^{k-2}$ to $\mathbb{R}P^{k-1} \to \mathbb{R}P^{k-2}$, and these homeomorphisms are obtained by from each other via precomposing the antipodal map of S^{k-1}, which has degree $(-1)^k$. Hence,

$$\deg(q\phi_k) = \deg(\text{id}) + \deg(\text{ant}) = 1 + (-1)^k = \begin{cases} 0 & k \text{ odd}, \\ 2 & k \text{ even}. \end{cases}$$

In the case of $\mathbb{R}P^4$, this leaves us with the CW complex

$$0 \to \mathbb{Z} \xrightarrow{d_4 \equiv 2} \mathbb{Z} \xrightarrow{d_3 \equiv 0} \mathbb{Z} \xrightarrow{d_2 \equiv 2} \mathbb{Z} \xrightarrow{d_1 \equiv 0} \mathbb{Z} \to 0.$$

(b) From the above, we note that for $k = 0, 1, 3$, $\ker(d_k) = \mathbb{Z}$ and $\mathrm{im}\,(d_k) = 0$, and for $k = 2, 4$, $\ker(d_k) = 0$ and $\mathrm{im}\,(d_k) = \mathbb{Z}\left\langle 2e^{k-1}\right\rangle$. Also note that $\mathrm{im}\,(d_5) = 0$. Thus,

$$H_k(X) = \frac{\ker(d_k)}{\mathrm{im}\,(d_{k+1})} = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z}/2\mathbb{Z} & k = 1, 3, \\ 0 & \text{otherwise.} \end{cases}$$

To compute $H_k(X, X^{(2)}; \mathbb{Z})$, we have the cellular complex

$$0 \to \frac{C_4(X)}{C_4(X^{(2)})} \to \frac{C_3(X)}{C_3(X^{(2)})} \to \frac{C_2(X)}{C_2(X^{(2)})} \to \frac{C_1(X)}{C_1(X^{(2)})} \to \frac{C_0(X)}{C_0(X^{(2)})} \to 0.$$

However, note that, by construction,

$$C_k(X^{(2)}) = \begin{cases} C_k(X) & 0 \leqslant k \leqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we obtain the CW structure

$$0 \to \mathbb{Z} \xrightarrow{d_4 \equiv 2} \mathbb{Z} \xrightarrow{d_3 \equiv 0} 0 \to 0 \to 0 \to 0.$$

Thus, we get

$$H_k(X, X^{(2)}) = \frac{\ker(d_k)}{\mathrm{im}\,(d_{k+1})} = \begin{cases} \mathbb{Z}/2\mathbb{Z} & k = 3, \\ 0 & \text{otherwise.} \end{cases}$$

■

Fall 2020, 8 List all the 3-sheeted connected covering spaces of $S^1 \vee S^1$. Which ones in the list are not normal?

One approach to classifying the 3-sheeted covering spaces is the use the Galois correspondence with index 3 subgroups of $\pi_1(S^1 \vee S^1)$, but because the group structure of $\mathbb{Z} * \mathbb{Z}$ is sufficiently complicated, it is actually more straight forward to find the covering spaces directly.

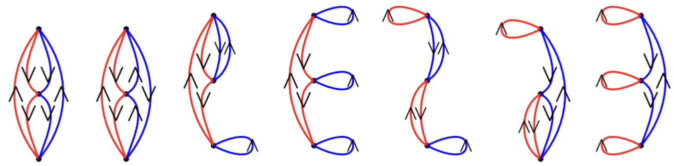
Our approach uses the fact that a covering space of a CW complex can be given a CW complex structure by lifting the characteristic maps to the covering space (see Hatcher Appendix: Topology of Cell Complexes Exercise 1). So in our case, a 3-sheeted connected covering corresponds to a connected graph with three vertices each of degree four; moreover if we label the edges of $S^1 \vee S^1$ as a and b, then each vertex of the covering space must have an incoming and an outgoing a edge, and an incoming and outgoing b edge.

For just one type of edges, i.e. either for a edges of b edges, there are three configurations that satisfy the directed degree requirement. Either a simple loop is attached to each vertex, there is one simple loop and one two cycle, or there is one three cycle. So we can combine these three types of orientations up to orientation, and with the requirement that the graph is connected.

First, there are two covering spaces formed with two three cycles, with the same and opposite orientation. When one edge type is given a three cycle, the other edge type may also be given three simple loops or one simple loop and one two cycle. This gives four possible covering spaces, based on which edge type has the three cycle; note that reversing orientation here gives the same graph.

And finally, both edge types can be given a two cycle and simple loop configuration. This gives one last covering space, and the orientation does not change the graph. All other combinations do not produce a connected graph, so these are all the covering spaces.

Finally, we can see that only the first two covering spaces are normal, because the group action there is transitive. Below is a visualization[3] of the covering spaces, the first two being the normal ones.



■

Fall 2020, 9 Let Σ_5 be a compact oriented surface of genus 5 without boundary. Does there exist an immersion $f : T^2 \to \Sigma_5$? Justify your answer.

Suppose such an f existed. As T^2 and Σ_5 are both 2-dimensional manifolds, f is then a local diffeomorphism between compact spaces, and as such[4] is a covering map. However, then $0 = \chi(T^2) = k\chi(\Sigma_5) = k(2 - 2 * 5) = -8k$, where k is the number of sheets of our covering map. This implies $k = 0$, which is impossible. ■

Fall 2020, 10 Show that the Euler characteristic of the special linear group $SL(n, \mathbb{R})$ with $n > 1$ is zero. Here for a topological space X its Euler characteristic is

$$\chi(X) := \sum_i (-1)^i \text{rank}(H_i(X)),$$

assuming that $\sum_i \text{rank}(H_i(X)) < \infty$.

We show that $SL_n(\mathbb{R})$ is homotopy equivalent to $SO(n)$, and since (clear from the definition given) Euler characteristic is invariant under homotopy, $\chi(SL_n(\mathbb{R})) = \chi(SO(n))$. Let $r : M_n(\mathbb{R}) \to SO(n)$ by $A = UP \mapsto U$, where UP is the polar decomposition of A so that P is positive definite and U is unitary and therefore $U \in SO(n)$. Let $i : SO(n) \hookrightarrow M_n(\mathbb{R})$ be the inclusion. By uniqueness of the polar decomposition, $i \circ r = id$, so we show that $r \circ i$ is homotopy equivalent to id. To do this, consider $H_t : SL_n(\mathbb{R}) \to SL_n(\mathbb{R})$ defined by $H_t(A) = \frac{(1-t)A+tU}{\det((1-t)A+tU)}$. We note $\det((1-t)A + tU) \neq 0$ as

$$(1-t)A + tU = U((1-t)P + tI).$$

As P is a positive definite matrix, and the convex combination of positive definite matrices is positive definite, and $\det(U) \neq 0$, we have that $\det((1-t)A + tU) \neq 0$. Additionally, note $t, \det(H_t(A)) = 1$. Moreover, we note that $H_0 = \text{id}$ and $H_1 = r \circ i$, as desired.

Since $SO(n)$ and $SL_n(\mathbb{R})$ are homotopy equivalent, they have the same Euler characteristic. Moreover, as $SO(n)$ is a lie group, it is parallelizable and thus admits a nowhere vanishing vector field.

[3]Taken from Prof. Rahul Panharipande's solutions here.

[4]Should prove this statement on exam.

Since $SO(n)$ is closed (it's the inverse image of $\{1\}$ for the map $A \mapsto AA^T$) and bounded, Poincare-Hopf implies $\chi(SO(n)) = 0$. ∎

15.2 Spring 2021

Spring 2021, 1 Without using homology groups or homotopy groups, directly derive Brouwer's fixed point theorem (any continuous map $f : D^2 \to D^2$ has a fixed point, where D^2 is the closed 2-disk) from the hairy ball theorem (any continuous vector field on S^2 is somewhere 0).

We did not figure this out however here is a solution from Wikipedia, which has yet to fail us. ∎

Spring 2021, 2 Solve the following problems:

(a) Let $F : S^n \to S^n$ be a continuous map. Show that if F has no fixed point, then the degree of the map, $\deg F = (-1)^{n+1}$.

(b) Show that if X has S^{2n} as universal covering space, then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

(a) We show this via Lefschetz theory. Since F has no fixed point, we know $L(F) = 0$. However,

$$\begin{aligned} 0 = L(F) &= \sum_{i=0}^{n} (-1)^i \mathrm{tr}(F_* : H_k(S^n) \to H_k(S^n)) \\ &= \mathrm{tr}(F_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \mathrm{tr}(F_* : H_n(S^n) \to H_n(S^n)) \\ &= 1 + (-1)^n \deg(F), \end{aligned}$$

implying our result.

(b) If S^{2n} is a universal covering space of X, then we know that $G(S^{2n}) \cong \pi_1(X)$, where $G(S^{2n})$ is the group of deck transformations of S^{2n}. If every deck transformation has a fixed point, then by the uniqueness of deck transformations, every deck transformation is simply the identity, so $\pi_1(X) = \{1\}$. If not, then by part (a), we know $\deg(F) = (-1)^{2n+1}$, which by the Poincare index theorem, implies F is homotopic to the antipodal map. Thus, we only have 2 unique deck transformations, implying $\pi_1(X) \cong \mathbb{Z}_2$.

∎

Spring 2021, 3 Let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{R}^3$. Calculate the integral homology groups of $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$.

If $n = 0$, then this is just the fundamental group of $\mathbb{R}^3$ which is trivial as $\mathbb{R}^3$ is simply connected. If $n = 1$, note that $\mathbb{R}^3 \setminus \{p\}$ deformation retracts onto S^2, which has a trivial fundamental group as that again is simply connected.

Suppose $n \geqslant 2$. Then, note that $\mathbb{R}^3 \setminus \{p_1, \ldots, p_n\}$ deformation retracts onto $S^2 \setminus \{q_1, \ldots, q_{n-1}\}$ for some $n-1$ distinct points in S^2, which is homotopy equivalent to $\mathbb{R}^2 - \{r_1, \ldots, r_{n-2}\}$ for some $n-2$ distinct points in $\mathbb{R}^2$.

Follow the same argument as in Fall 2022, 7 to get the fundamental group is a free product of $n-2$ copies of $\mathbb{Z}$. ∎

Spring 2021, 4 Let $\Delta_n^{(k)}$ be the k-dimensional skeleton of the n-simplex Δ_n. Calculate the reduced homology groups $\tilde{H}_i(\Delta_n^{(k)})$ for all values of i, k, n.

We first note that, as Δ_n is contractible, $\tilde{H}_i(\Delta_n) = 0$ for all i.

Note that for $i < k$

$$\tilde{H}_i(\Delta_n^{(k)}) = \tilde{H}_i(\Delta_n) = 0,$$

and for $i > k$, since there are no i-cells, $\tilde{H}_i(\Delta_n^{(k)}) = 0$. It suffices to find $\tilde{H}_k(\Delta_n^{(k)})$. Note, however, the top homology is always torsion-free (and thus free) since the previous boundary map is always 0. Thus, noting that

$$\chi(H_i(\Delta_n^{(k)})) = \sum_{i=0}^{k}(-1)^k \operatorname{rank}(H_k(\Delta_n^{(k)})) = 1 + (-1)^k \operatorname{rank}(H_k(\Delta_n^{(k)}))$$

and

$$\begin{aligned}\chi(H_i(\Delta_n^{(k)})) = \sum_{i=0}^{k}(-1)^i \# e_i = \sum_{i=0}^{k}(-1)^i\binom{n+1}{i+1} &= 1 - \sum_{i=0}^{k+1}(-1)^i\binom{n+1}{i}, \\ &= 1-(-1)^{k+1}\binom{n}{k+1} = 1+(-1)^k\binom{n}{k+1},\end{aligned}$$

we have

$$\operatorname{rank}(H_k(\Delta_n^{(k)})) = \binom{n}{k+1},$$

implying our result.

The combinatorial identity can be proven inductively and the number of i-cells in Δ_n is precisely number of size $i+1$ subsets of $\{0, \dots, n\}$, which is $\binom{n+1}{i+1}$. ∎

Alternate Solution: We establish

$$\widetilde{H}_i(\Delta_n^{(k)}) \cong \begin{cases} \mathbb{Z}^{\binom{n}{k+1}} & \text{if } i = k, \\ 0 & \text{otherwise} \end{cases}$$

for all i and $0 \leqslant k \leqslant n$ by induction on n. The base case $n = 0$ is clear. Now fix $n > 1$. Since $\Delta_n^{(0)}$ is the disjoint union of $n+1$ points, clearly

$$\widetilde{H}_i(\Delta_n^{(0)}) \cong \begin{cases} \mathbb{Z}^n & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}.$$

Now suppose $k > 1$. Since $\Delta_n^{(k)}$ is the mapping cone of the inclusion $\Delta_{n-1}^{(k-1)} \hookrightarrow \Delta_n^{(k)}$, we have a long exact sequence

$$\cdots \to \widetilde{H}_i(\Delta_{n-1}^{(k-1)}) \to \widetilde{H}_i(\Delta_{n-1}^{(k)}) \to \widetilde{H}_i(\Delta_n^{(k)}) \to \widetilde{H}_{i-1}(\Delta_{n-1}^{(k-1)}) \to \widetilde{H}_{i-1}(\Delta_{n-1}^{(k)}) \to \cdots$$

By induction this gives $\widetilde{H}_i(\Delta_n^{(k)}) = 0$ when $i \neq k$, implying we obtain a split short exact sequence, thus

$$\widetilde{H}_k(\Delta_n^{(k)}) \cong \mathbb{Z}^{\binom{n-1}{k}} \oplus \mathbb{Z}^{\binom{n-1}{k+1}} \cong \mathbb{Z}^{\binom{n}{k+1}}.$$

∎

Spring 2021, 5 Define the complex projective space $\mathbb{C}P^n$ to be the quotient of $\mathbb{C}^{n+1}\backslash\{0\}$ by the relation $x \sim \lambda x$ for all $\lambda \in \mathbb{C}\backslash\{0\}, x \in \mathbb{C}^{n+1}\backslash\{0\}$. Construct a CW complex structure on $\mathbb{C}P^n$ with no odd-dimensional cells and exactly 1 cell in each even dimension up to $2n$. Calculate the fundamental group and the integral homology groups of $\mathbb{C}P^n$.

We define a CW structure of $\mathbb{C}P^n$ which has 1 k-cell for each even k between 0 and $2n$. Our cell complex will then be

$$\cdots \to 0 \to 0 \to \mathbb{Z} \to 0 \to \cdots \to 0 \to \mathbb{Z} \to 0 \to 0 \to \dots$$

where the first $\mathbb{Z}$ is in grading $2n$ and the last is in grading 0, and we alternate between 0 and $\mathbb{Z}$ in between these. As all of the boundary maps are 0, the integral homology groups of $\mathbb{C}P^n$ are the same as the chain groups. Thus,

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & 0 \leqslant k \leqslant 2n,\ k \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

We also note that the fundamental group of a CW complex is completely determined by its 2-skeleton. In fact, it is a free group, with the 1-cells as the generators and the attaching maps of the 2-cells as the relations. As our construction of $\mathbb{C}P^n$ has no 1-cell, our fundamental group is trivial.

We now define our CW structure inductively. Note that $\mathbb{C}P^0$ is simply a point, so we just have a 0-cell. Given $\mathbb{C}P^{n-1}$, we claim we can construct $\mathbb{C}P^n$ by attaching a $2n$-cell. To see this, note that $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$, via the inclusion $[z_0, \dots, z_{n-1}] \mapsto [z_0, \dots, z_{n-1}, 0]$. Additionally note that an arbitrary point in $\mathbb{C}P^n - \mathbb{C}P^{n-1}$ can be represented by $(z_0, \dots, z_{n-1}, t)$, where $t > 0$ is the real number $\sqrt{1 - \sum z_i \overline{z_i}}$. This defines a map

$$e^{2n} \to \mathbb{C}P^n : z = (z_0, \dots, z_{n-1}) \mapsto [z_0, \dots, z_{n-1}, t],$$

with $t = \sqrt{1 - \sum z_i \overline{z_i}}$. The boundary of e^{2n} (where $t = 0$), is sent to $\mathbb{C}P^{n-1}$. ∎

Spring 2021, 6 Define the orientation double cover for any topological manifold. What is the orientation double cover of the real projective plane $\mathbb{R}P^n$?

The orientation double cover for any topological manifold is the unique two-fold orientable covering space of a topological manifold with orientation reversing non-trivial deck transformation.

We first note that $\mathbb{R}P^n$ is the quotient space of S^n by the antipodal map, and that

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & k = 0,\ k = n \text{ and } n \text{ odd}, \\ \mathbb{Z}/2\mathbb{Z} & 1 \leqslant k < n,\ k \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

So, $\mathbb{R}P^n$ is orientable if and only if n is odd. We then see that the orientation cover, in this case, is $\mathbb{R}P^n \sqcup \mathbb{R}P^n$, with opposite orientations on the two copies. So, the non-trivial deck transformation taking $x_1 \mapsto x_2$, where $p(x_1) = x = p(x_2)$ is orientation reversing.

When n is even, we claim the orientation double cover is S^n, however this is clear as S^n is an orientable double cover of $\mathbb{R}P^n$, and the non-trivial deck transformation is the antipodal map A, where $x \mapsto -x$. When n is even, we note that $\deg(A) = (-1)^{n+1} = -1$, as A is the composition of $n + 1$ negations, all of which being orientation reversing diffeomorphisms. So the non-trivial deck transformation is orientation reversing. ∎

Spring 2021, 7 Show that $S^2 \times S^2$ and the connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ are not homotopy equivalent.

We will show that these two spaces have different ring structure.

First, we will show that for closed oriented manifolds M and N of dimension n,

$$\widetilde{H}_i(M\#N) = \begin{cases} \widetilde{H}_i(M) \oplus \widetilde{H}_i(N), & i < n; \\ \mathbb{Z}, & i = n. \end{cases}$$

To show this, consider $S^{n-1} \subset M\#N =: X$ to be the disk that M and N are glued by. Notice that then (X, S^{n-1}) is a good pair and $X/S^{n-1} \simeq M \vee N$. Therefore we get an exact sequence of reduced homologies:

$$\ldots \to \widetilde{H}_i(S^{n-1}) \to \widetilde{H}_i(X) \to \widetilde{H}_i(M \vee N) \to \ldots$$

Since $H_i(S^{n-1}) = 0$ for $i < n-1$, we get an isomorphism

$$\widetilde{H}_i(X) \simeq \widetilde{H}_i(M \vee N) \simeq \widetilde{H}_i(M) \oplus \widetilde{H}_i(N).$$

Now, for the remaining homologies we get the exact sequence:

$$\begin{aligned} 0 \to \widetilde{H}_n(X) \simeq \mathbb{Z} \to \widetilde{H}(M \vee N) \simeq \mathbb{Z}^2 \to \widetilde{H}_{n-1}(S^{n-1}) \simeq \mathbb{Z} \\ \to \widetilde{H}_{n-1}(X) \to \widetilde{H}_{n-1}(M \vee N) \simeq \widetilde{H}_{n-1}(M) \oplus \widetilde{H}_{n-1}(N) \to 0 \end{aligned}$$

Notice that since M, N and X are all closed orientable, their H_{n-1} is torsion free. Moreover, computing the alternating sum of ranks, we get that $\widetilde{H}_{n-1}(X)$ and $H_{n-1}(M) \oplus H_{n-1}(N)$ have the same rank (and are both free!). Therefore, they are isomorphic.

Now notice that in the case of $\mathbb{C}P^n$, all homology groups are free, so the cohomology groups are dual to them. That is,

$$H^i(\mathbb{C}P^2 \# \mathbb{C}P^2) = \begin{cases} \mathbb{Z}^2, & i = 2, \\ \mathbb{Z}, & i = 0, 4, \\ 0 & \text{otherwise.} \end{cases}$$

That means that the generators of the two copies of $H^4(\mathbb{C}P^2)$ are identified in $H^4(\mathbb{C}P^2 \# \mathbb{C}P^2)$.

Thus, letting α_1, α_2 be generators of $H^2(\mathbb{C}P^2 \# \mathbb{C}P^2)$, we get

$$\begin{aligned} \alpha_1 \smile \alpha_1 &= \alpha_2 \smile \alpha_2 \\ \alpha_1 \smile \alpha_2 &= 0 = \alpha_2 \smile \alpha_1 \end{aligned}$$

where $\alpha_1 \smile \alpha_1 = \alpha_2 \smile \alpha_2$ generates $H^4(\mathbb{C}P^2 \# \mathbb{C}P^2)$. The matrix corresponding to this structure is the 2×2 identity matrix, which is positive definite. Meanwhile, via the Künneth Formula, letting θ be a volume form in S^2, we have $\pi_1 * (\theta)$ and $\pi_2^*(\theta)$ the generators of $H^2(S^2 \times S^2)$, and

$$\begin{aligned} &\pi_1^*(\theta) \smile \pi_1^*(\theta) = 0 \\ &\pi_1^*(\theta) \smile \pi_2^*(\theta) = (-1)^4(\pi_2^*(\theta) \smile \pi_1^*(\theta)) = \pi_2^*(\theta) \smile \pi_1^*(\theta) \\ &\pi_2^*(\theta) \smile \pi_2^*(\theta) = 0, \end{aligned}$$

where $\pi_1^*(\theta) \smile \pi_2^*(\theta)$ generates $H^4(S^2 \times S^2)$. The matrix corresponding to this is the 2×2 nontrivial permutation matrix, which is not positive definite as it has a positive and negative eigenvalue.

Thus, the two cup product structures are not the same and thus the two cohomology rings are not isomorphic. ∎

Spring 2021, 8 Consider a differentiable map $f : S^{2n-1} \to S^n$ with $n \geqslant 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n such that $\int_{S^n} \alpha = 1$, let $f^*\alpha \in \Omega^n(S^{2n-1})$ be its pull-back under f.
(a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.
(b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choices of β and α.

(a) We wish to show that $f^*(\alpha)$ is an exact form in $\Omega^n(S^{2n-1})$. However, as α is an n-form, we have that $d\alpha = 0$, so α is closed. This implies that $f^*(\alpha)$ is closed as

$$df^*(\alpha) = f^*(d\alpha) = f^*(0) = 0.$$

Note that as $n \geqslant 2$, we have that $n < n + n - 1 = 2n - 1$. So, as $H^n(S^{2n-1}) = 0$ and $f^*(\alpha)$ is closed, we have that $f^*(\alpha)$ is exact, thus there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

(b) We first show $I(f)$ is independent of the choice of β. That is, suppose $\beta, \beta' \in \Omega^{n-1}(S^{2n-1})$ such that $d\beta' = f^*(\alpha) = d\beta$. Then, $d(\beta' - \beta) = f^*\alpha - f^*\alpha = 0$, so $\beta' - \beta$ is closed. As $H^{n-1}(S^{2n-1}) = 0$, this implies $\beta' - \beta$ is exact, so $\beta' - \beta = d\gamma$ for some $\gamma \in \Omega^{n-2}(S^{2n-1})$. Thus,

$$\int_{S^{2n-1}} \beta' \wedge d\beta' = \int_{S^{2n-1}} (\beta + d\gamma) \wedge d(\beta + d\gamma) = \int_{S^{2n-1}} \beta \wedge d\beta + \int_{S^{2n-1}} d\gamma \wedge d\beta.$$

We are done if we show the last integral vanishes. However, by Stoke's

$$\int_{S^{2n-1}} d\gamma \wedge d\beta = \int_{S^{2n-1}} d(\gamma \wedge d\beta) = \int_{\partial S^{2n-1}} d^2(\gamma \wedge d\beta) = 0.$$

Now suppose $\alpha, \alpha' \in \Omega^n(S^n)$ are such that $\int_{S^n} \alpha = 1 = \int_{S^n} \alpha'$. Then, as $H^n(S^n) = \mathbb{Z}$, we have $\alpha = \alpha' + d\eta$ for some $\eta \in \Omega^{n-1}(S^{n-1})$. Let $\beta, \beta' \in \Omega^{n-1}(S^{2n-1})$ be such that $d\beta = f^*(\alpha)$ and $d\beta' = f^*(\alpha')$. Then,

$$d\beta = f^*(\alpha) = f^*(\alpha' + d\eta) = f^*(\alpha') + df^*(\eta) = d\beta' + d\eta' = d(\beta' + \eta').$$

for some $\eta' \in \Omega^{n-1}(S^{2n-1})$. Thus, $\beta - \beta' - \eta'$ is closed, implying $\beta = \beta' + \eta' + d\gamma$ for some $\gamma \in \Omega^{n-2}(S^{2n-1})$. Thus,

$$\begin{aligned}\int_{S^{2n-1}} \beta \wedge d\beta - \beta' \wedge d\beta' &= \int_{S^{2n-1}} (\beta' + \eta' + d\gamma) \wedge (d\beta' + d\eta') - \beta' \wedge d\beta' \\ &= \int_{S^{2n-1}} \beta' \wedge d\eta' + \eta' \wedge d\beta' + \eta' \wedge d\eta' + d\gamma \wedge d\beta' + d\gamma \wedge d\eta' \\ &= \int_{S^{2n-1}} d(\beta' \wedge \eta') + \int_{S^{2n-1}} \eta' \wedge d\eta' + \int_{S^{2n-1}} d(\gamma \wedge d\beta' + \gamma \wedge d\eta').\end{aligned}$$

Note that $\eta' \wedge d\eta' = f^*(\eta) \wedge df^*(\eta) = f^*(\eta \wedge d\eta) = f^*(0) = 0$. Thus, as these integrals are all of exact forms, they integrate to 0 via Stoke's.

■

Spring 2021, 9 Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that

$$f^*(L_Y\omega) = L_X(f^*\omega).$$

We first show $f^* i_Y \omega = i_X f^* \omega$ for any $\omega \in \Omega^k(M)$. Let $E_1, \ldots, E_{k-1}$ be vector fields on M. Then,

$$
\begin{aligned}
f^* i_Y \omega(E_1, \ldots, E_{k-1}) = i_Y \omega(f_* E_1, \ldots, f_* E_{k-1}) &= \omega(Y, f_* E_1, \ldots, f_* E_{k-1}) \\
&= \omega(f_* X, f_* E_1, \ldots, f_* E_{k-1}) \\
&= f^* \omega(X, E_1, \ldots, E_{k-1}) \\
&= i_X f^* \omega(E_1, \ldots, E_{k-1}).
\end{aligned}
$$

Thus, as $df^* = f^* d$, we have, via Cartan's magic formula

$$
\begin{aligned}
f^*(L_Y \omega) = f^*((di_Y + i_Y d)\omega) = f^* di_Y \omega + f^* i_Y d\omega = di_X f^* \omega + i_X df^* \omega& \\
= (di_X + i_X d)(f^* \omega) = L_X(f^* \omega).&
\end{aligned}
$$

■

Spring 2021, 10 Prove Cartan's lemma: Let M be a smooth manifold of dimension n. Fix $1 \leqslant k \leqslant n$. Let ω^i and φ_i be 1-forms on M. Suppose that the $\{\omega^1, \ldots, \omega^k\}$ are linearly independent and that $\sum_{i=1}^k \varphi_i \wedge \omega^i = 0$. Prove that there exist smooth functions $h_{ij} = h_{ji} : M \to \mathbb{R}$ such that for all $i = 1, \ldots, k$, $\varphi_i = \sum_{j=1}^k h_{ij} \omega^j$.

We first prove that $\{\alpha_1, \ldots, \alpha_r\}$ is a linearly dependent set of 1-forms if and only if $\alpha_1 \wedge \cdots \wedge \alpha_r = 0$. The forward direction is clear, by linearity of wedge product and $\alpha_i \wedge \alpha_i = 0$ as α_i is a 1-form. Now let $\alpha_1, \ldots, \alpha_r$ be a linearly independent set of 1-forms. Let $p \in M$ be given and consider a dual basis $X_1, \ldots, X_k$ to $\alpha_1, \ldots, \alpha_r$. Then, $\alpha_1 \wedge \cdots \wedge \alpha_1(X_1, \ldots, X_k) = 1$, implying $\alpha_1 \wedge \cdots \wedge \alpha_r \neq 0$.

Fix some $1 \leqslant j \leqslant k$. First note that by linearity of the wedge product, we obtain

$$
(\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k) \wedge \left(\sum_{i=1}^k \phi_i \wedge \omega^i \right) = (\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k) \wedge 0 = 0,
$$

where $\widehat{\omega}^j$ indicates omitting ω^j. However, expanding this, we also obtain

$$
\begin{aligned}
(\omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k) \wedge \left(\sum_{i=1}^k \phi_i \wedge \omega^i \right) &= \omega^1 \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k \wedge \phi_j \wedge \omega^j \\
&= \pm \omega^1 \wedge \cdots \wedge \omega^k \wedge \phi_j.
\end{aligned}
$$

Thus,

$$
\omega^1 \wedge \cdots \wedge \omega^k \wedge \phi_j = 0.
$$

This, however, implies that $\{\omega^1, \ldots, \omega^k, \phi_j\}$ is a linearly dependent set, thus

$$
\phi_j = \sum_{j=1}^k h_{ij} \omega^j.
$$

Since each ω^i is smooth and ϕ_j is smooth, each h_{ij} must be smooth as well. We now show $h_{ij} = h_{ji}$.

Note that, fixing $1 \leqslant i < j \leqslant k$, we have

$$
0 = \left(\sum_{r=1}^k \phi_r \wedge \omega^r \right) \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k
$$

$$= \phi_i \wedge \omega^i \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k + \phi_j \wedge \omega^j \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k$$
$$= h_{ij}\omega^j \wedge \omega^i \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k + h_{ji}\omega^i \wedge \omega^j \wedge \omega^1 \wedge \cdots \wedge \widehat{\omega}^i \wedge \cdots \wedge \widehat{\omega}^j \wedge \cdots \wedge \omega^k$$

However this implies
$$h_{ij}\omega^1 \wedge \cdots \wedge \omega^k = h_{ji}\omega^1 \wedge \ldots \omega^k.$$
As $\{\omega^1, \ldots, \omega^k\}$ is a linearly independent set, we have $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$, thus we must have $h_{ij} = h_{ji}$, as desired. ■

15.3 Fall 2021

Fall 2021, 1 Let $V_k(\mathbb{R}^n)$ denote the space of k-tuples of orthonormal vectors in $\mathbb{R}^n$. Show that $V_k(\mathbb{R}^n)$ is a manifold of dimension $k\left(n - \frac{k+1}{2}\right)$. Hint: Use a map $F : M_{n\times k}(\mathbb{R}) \to \mathbb{R}^{k(k+1)/2}$ such that $V_k(\mathbb{R}^n))$ becomes the preimage of a regular value of F. (Here $M_{n\times k}(\mathbb{R})$ denotes the set of matrices with n rows and k columns.

Consider the map $F : M_{n\times k}(\mathbb{R}) \to \mathrm{Sym}(M_k(\mathbb{R}))$ given by $A \mapsto A^T A$. We note that $V_k(\mathbb{R}^n) = F^{-1}(I)$. Letting $C \in \mathrm{Sym}(M_k(\mathbb{R}))$ and having $B = \frac{1}{2}AC$, we note that

$$dF_A(B) = \lim_{t\to 0} \frac{1}{t}\left(F(A+tB) - F(A)\right) = B^T A + A^T B = \frac{1}{2}\left(C^T A^T\right) A + \frac{1}{2}A^T A C = \frac{1}{2}C^T + \frac{1}{2}C = C,$$

as $C^T = C$. Thus I is a regular value of F, implying $F^{-1}(I)$ is a manifold of dimension

$$\dim(M_{n\times k}) - \dim(\mathrm{Sym}(M_k(\mathbb{R}))) = kn - \frac{k(k+1)}{2} = k\left(n - \frac{k+1}{2}\right).$$

■

Fall 2021, 2 Show that the product of two spheres $S^p \times S^q$ is parallelizable provided p or q is odd. (Here parallelizable means the tangent bundle is trivializable; equivalently, there exist $(p+q)$ vector fields on $S^p \times S^q$ which are everywhere linearly independent.)

WLOG suppose p is odd. Define the map $X : S^p \to TS^p$ with

$$x \mapsto (-x_2, x_1, -x_4, x_3, \ldots, -x_{p+1}, x_p).$$

We note that X is a vector field as, as in fact

$$\begin{aligned} x \cdot X(x) &= (x_1, x_2, x_3, x_4, \ldots, x_p, x_{p+1}) \cdot (-x_2, x_1, -x_4, x_3, \ldots, -x_{p+1}, x_p) \\ &= -x_1x_2 + x_2x_1 - x_4x_3 + x_3x_4 - \cdots - x_px_{p+1} + x_{p+1}x_p = 0, \end{aligned}$$

Define the bundle B where $B_x = \mathrm{span}\,(X(x))$. Then, B is a one-dimensional bundle and thus $B \cong S^p \times \mathbb{R}$. We also note that $TS^p = B \oplus B^\perp$. We also note that the normal bundle NS^p is trivial as S^p has codimension one when embedded into R^{p+1}. Additionally, we note, for $x \in S^p$,

$$T\mathbb{R}^{p+1}_x = TS^p_x + NS^p_x.$$

The same is true for S^q. Additionally, letting $\pi_1 : S^p \times S^q \to S^p$ and $\pi_2 : S^p \times S^q \to S^q$ be the canonical projections, we have

$$\begin{aligned}
T(S^p \times S^q) = \pi_1^*(TS^p) \oplus \pi_2^*(TS^q) &= \pi_1^*(B \oplus B^\perp) \oplus \pi_2^*(TS^q) \\
&= \pi_1^*(B^\perp) \oplus S^p \times \mathbb{R} \oplus \pi_2^*(TS^q) \\
&= \pi_1^*(B^\perp) \oplus \pi_2^*(TS^q \oplus S^q \times \mathbb{R}) \\
&= \pi_1^*(B^\perp) \oplus \pi_2^*(TS^q \oplus NS^q) \\
&= \pi_1^*(B^\perp) \oplus \pi_2^*(S^q \times \mathbb{R}^{q+1}) \\
&= \pi_1^*(B^\perp \oplus S^q \times \mathbb{R}^2) \oplus S^p \times S^q \times \mathbb{R}^{q-1} \\
&= \pi_1^*(TS^p \oplus NS^p) \oplus S^p \times S^q \times \mathbb{R}^{q-1} \\
&= S^p \times S^q \times \mathbb{R}^{p+1} \oplus S^p \times S^q \times \mathbb{R}^{q-1} = S^p \times S^q \times \mathbb{R}^{p+q}.
\end{aligned}$$

Thus $S^p \times S^q$ is parallelizable.

■

Fall 2021, 3 Let $M^m \subset \mathbb{R}^n \backslash \{0\}$ be a compact smooth submanifold of dimension m. Show that M is transverse to almost all k-dimensional linear subspaces in $\mathbb{R}^n$. (Here "almost all" means that the set of subspaces that are not transverse to M has measure zero.)

Let $U \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$ denote the open set of linearly independent vectors in $(\mathbb{R}^n)^k$. Consider the map $F : (\mathbb{R}^k \backslash \{0\}) \times U \to \mathbb{R}^n$ via

$$(\alpha^1, \dots, \alpha^k, v_1, \dots, v_k) \mapsto \sum_i \alpha^i v_i.$$

Note that we may restrict F to $\mathbb{R}^n \backslash \{0\}$ as $\sum_i \alpha^i v_i = 0$ if and only if $\alpha^i = 0$ for all i as this set of vectors are linearly independent. We claim that F is a submersion.

Note that given $(\alpha^1, \dots, \alpha^k, v_1, \dots, v_k)$, there exists some i such that $\alpha^i \neq 0$, thus for any $h \in \mathbb{R}^n$

$$\begin{aligned}
dF_{(\alpha^1,\dots,\alpha^k,v_1,\dots,v_k)}(0, \dots, 0, 0, \dots, h, \dots, 0) = \lim_{t\to 0} \frac{1}{t}(&F(\alpha^1, \dots, \alpha^k, v_1, \dots, v_i + h, \dots, v_k) \\
&- F(\alpha^1, \dots, \alpha^k, v_1, \dots, v_k)) = \alpha^i h.
\end{aligned}$$

As F is a submersion, F is transverse to $M \subset \mathbb{R}^n \backslash \{0\}$. Thus, by Thom's transversality theorem, for almost all $v = (v_1, \dots, v_k) \in U$ the map $F_v : \mathbb{R}^k \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is transverse to M. This implies that the k-dimensional linear subspace spanned by the set v is transverse to M for almost all linearly independent sets of k vectors. As this is true for almost all linearly independent sets of k vectors, this is true for almost all k-dimensional linear subspaces, as there is a surjection $P : U \to Gr(k, n)$, where $Gr(k, n)$ denotes the space of k-dimensional linear subspaces of $\mathbb{R}^n$. ■

Fall 2021, 4 Let $\omega \in \Omega_c^n(\mathbb{R}^n)$ be a compactly supported n-form. Show that $\omega = d\eta$ for some compactly supported $(n-1)$-form $\eta \in \Omega_c^{n-1}(\mathbb{R}^n)$ if and only if $\int_{\mathbb{R}^n} \omega = 0$.

($\to$) Suppose $\omega = d\eta$. Then
$$\int_{\mathbb{R}^n} \omega = \int_{\partial \mathbb{R}^n} \eta = 0$$
as η has compact support and $\partial \mathbb{R}^n = \varnothing$, so we may apply Stokes theorem on compactly supported form.

($\leftarrow$) Suppose $\int_{\mathbb{R}^n} \omega = 0$. There are two cases.

Case 1: $n = 1$. Then $\omega = f dx$ for some compactly supported $f \in C^\infty(\mathbb{R})$. Define $F : \mathbb{R} \to \mathbb{R}$ as
$$F(x) = \int_{-\infty}^{x} f(t)\, dt.$$
By the fundamental theorem of calculus, $dF = F'\, dx = f\, dx = \omega$. As f is compactly supported, there exists some $R > 0$ such that $\operatorname{supp} f \subseteq [-R, R]$. When $x < -R$, $F(x) = 0$ and when $x > R$, as $\int_{\mathbb{R}} \omega = 0$, we have
$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt = 0.$$
Thus, $\operatorname{supp} F \subseteq [-R, R]$.

Case 2: $n > 1$. Let $B, B' \in \mathbb{R}^n$ be open balls centered at the origin such that $\operatorname{supp} \omega \subset B \subset \overline{B} \subset B'$. By Poincare's Lemma , for every closed form ω, there exists a smooth $(n-1)$-form τ on $\mathbb{R}^n$ such that $d\tau = \omega$. In particular, we note $d\tau = 0$ on $\mathbb{R}^n - \overline{B}$.

Consider the restriction of τ to $\mathbb{R}^n - \overline{B}$. Note that τ is closed on this domain. Additionally note that the inclusion $\iota : S \to \mathbb{R}^n - \overline{B}$ induces an isomorphism $\iota^\star : H^{n-1}_{dR}(\mathbb{R}^n - \overline{B}) \to H^{n-1}_{dR}(S)$ as $\mathbb{R}^n - \overline{B}$ deformation retracts onto S, some $(n-1)$-sphere contained in $\mathbb{R}^n - \overline{B}$ centered at the origin. However, it then follows, since τ is closed on $\mathbb{R}^n - \overline{B}$, that τ is exact on $\mathbb{R}^n - \overline{B}$ if and only if $\iota^\star \tau$ is exact on S, which in turn is true if and only if $\int_S \iota^\star \tau = 0$. However, Stokes implies
$$0 = \int_{\mathbb{R}^n} \omega = \int_{\overline{B'}} \omega = \int_{\overline{B'}} d\tau = \int_{\partial \overline{B'}} \tau.$$
Thus, τ is exact on $\mathbb{R}^n - \overline{B}$.

Ergo, there is a smooth $(n-2)$-form γ on $\mathbb{R}^n - \overline{B}$ such that $\tau = d\gamma$. Letting ϕ be a smooth bump function that is supported on $\mathbb{R}^n - \overline{B}$ and equal to 1 on $\mathbb{R}^n - B'$, we have:

- $\eta = \tau - d(\phi\gamma)$ is smooth on all of $\mathbb{R}^n$,
- $d\eta = d\tau - d^2(\phi\gamma) = d\tau = \omega$,
- η is compactly supported.

■

Fall 2021, 5 Let $n \geqslant 0$ be an integer. Let M be a compact, orientable, smooth manifold of dimension $4n + 2$. Show that $\dim H^{2n+1}(M; \mathbb{R})$ is even.

See Spring 2015 #10. Idea: Use Poincaré Duality to define an alternating non-degenarate bilinear form. ■

Fall 2021, 6 Let $f : \mathbb{C} \to \mathbb{C}$ be a nowhere zero continuous function. Prove that there exists a continuous function $g : \mathbb{C} \to \mathbb{C}$ such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$.

As f is nowhere zero, we may restrict f to a map $f : \mathbb{C} \to \mathbb{C}\backslash\{0\}$. Noting that the map $\exp : \mathbb{C} \to \mathbb{C}\backslash\{0\}$ is a covering map, and that $f_*(\pi_1(\mathbb{C}, \star)) = f_*(0) = 0 = \exp_*(0) = \exp_*(\pi_1(\mathbb{C}, \star))$ as $\mathbb{C}$ is contractible, there exists a lift $g : \mathbb{C} \to \mathbb{C}$ such that $f = \exp \circ g$, as desired. ∎

Fall 2021, 7 In this problem, work in either the category of topological manifolds or smooth manifolds (your choice). Let M be an n-manifold. Define its orientation double cover $\tilde{M}$, and explain its structure as a topological/smooth manifold. Prove that the orientation double cover of $\tilde{M}$ is always disconnected.

We work in the category of smooth manifolds.

Define the orientation cover $\tilde{M}$ as the set of orientations of all tangent spaces to M:

$$\tilde{M} = \{(p, \mathcal{O}_p) \mid p \in M \text{ and } \mathcal{O}_p \text{ is the orientation of } T_pM\}.$$

Define the projection $\pi : \tilde{M} \to M$ where $(p, \mathcal{O}_p) \mapsto p$. Since each tangent space has exactly two orientations, the fiber of this map has cardinality 2.

For each pair $(U, \mathcal{O})$, where U is an open subset of M and $\mathcal{O}$ is an orientation on U, define a subset $\tilde{U}_{\mathcal{O}} \subset \tilde{M}$ as follows:

$$\tilde{U}_{\mathcal{O}} = \{(p, \mathcal{O}_p) \mid p \in U \text{ and } \mathcal{O}_p \text{ is the orientation of } T_pM \text{ determined by } \mathcal{O}\}.$$

We show that the collection of all subsets of the form $\tilde{U}_{\mathcal{O}}$ is a basis for a topology on $\tilde{M}$. Given some $(p, \mathcal{O}_p) \in \tilde{M}$, let U be an orientable neighborhood of p in M, and let $\mathcal{O}$ be some orientation on it. We may assume $\mathcal{O}_p$ is the same as the orientation of T_pM determine by $\mathcal{O}$ by replacing $\mathcal{O}$ with $-\mathcal{O}$ if necessary. It follows that $(p, \mathcal{O}_p) \in \tilde{U}_{\mathcal{O}}$, so the collection of these sets cover $\tilde{M}$.

If $\tilde{U}_{\mathcal{O}}$ and $\tilde{U}'_{\mathcal{O}'}$ are two such sets and $(p, \mathcal{O}_p)$ are in their intersection then $\mathcal{O}_p$ is determined by both $\mathcal{O}$ and $\mathcal{O}'$. If V is the component of $U \cap U'$ containing p, then the restricted orientations $\mathcal{O}|_V$ and $\mathcal{O}'|_V$ agree at p. Since these two orientations agree at a point, they agree on all of V. Thus, $\tilde{U}_{\mathcal{O}} \cap \tilde{U}'_{\mathcal{O}'}$ contains the basis set $\tilde{V}_{\mathcal{O}|_V}$. We thus have a topology.

To prove the orientation double cover of $\tilde{M}$ is always disconnected, we first prove $\tilde{M}$ is orientable. Let $\tilde{p} = (p, \mathcal{O}_p)$ be a point in $\tilde{M}$. By definition, $\mathcal{O}_p$ is an orientation on T_pM, so we can give $T_{\tilde{p}}\tilde{M}$ the unique orientation $\tilde{\mathcal{O}}_{\tilde{p}}$ so that $d\pi_{\tilde{p}} : T_{\tilde{p}}\tilde{M} \to T_pM$ is orientation-preserving. This defines a pointwise orientation $\tilde{\mathcal{O}}$ on $\tilde{M}$. On each basis open subset $\tilde{U}_{\mathcal{O}}$ the orientation $\tilde{\mathcal{O}}$ agrees with the pullback orientation induced from $(U, \mathcal{O})$ by (the restriction of) π, so it is continuous.

As $\tilde{M}$ is orientable, we note that $\tilde{M}$ is evenly covered by π, as every connected open orientable subset is evenly covered by π. This implies $\tilde{\tilde{M}}$ has two components, and thus is disconnected. ∎

Fall 2021, 8 Let M be a connected non-orientable manifold whose fundamental group G is simple (that is, has no non-trivial normal subgroup). Prove that G must be isomorphic to $\mathbb{Z}/2$

Consider the orientation double cover $\tilde{M} \xrightarrow{p} M$. As M is non-orientable, $\tilde{M}$ is connected, and as $\tilde{M}$ is a double cover, we have that $p_*(\pi_1(\tilde{M}, \star))$ is an index 2 subgroup of $\pi_1(M, \star) = G$. As all index 2 subgroups are normal, this implies, as G is simple, $p_*(\pi_1(\tilde{M}, \star)) = 0$. However the only group with a trivial index 2 subgroup is $\mathbb{Z}/2$. ∎

Fall 2021, 9 Let X be the quotient of the space $\{0,1,2\} \times S^1 \times D^2$ by the relation

$$(0, z_1, z_2) \sim (1, z_1, z_2) \sim (2, z_1, z_2)\, \forall z_1, z_2 \in S^1.$$

(Here S^1 is the unit circle and D^2 is the unit disk, both inside $\mathbb{R}^2$.) Compute the homology groups of X with integer coefficients.

We construct a Mayer-Vietoris LES. Let $A = \{0,1,2\} \times S^1 \times \{x \in D^2 \,|\, |x| > \frac{1}{3}\}/\sim$ and $B = \{0,1,2\} \times S^1 \times \{x \in D^2 \,|\, |x| < \frac{2}{3}\}/\sim$. Then, A deformation retracts onto $(\{0,1,2\} \times S^1 \times S^1/\sim) = S^1 \times S^1$, B deformation retracts onto $\{0,1,2\} \times S^1$, and $A \cap B$ deformation retracts onto $\{0,1,2\} \times S^1 \times S^1$.

We note that $H_*(S^1) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}_{(1)}$ and, via Künneth, $H_*(S^1 \times S^1) = \mathbb{Z}_{(0)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(2)}$. Thus,

$$\begin{aligned} H_*(A) &= \mathbb{Z}_{(0)} \oplus \mathbb{Z}^2_{(1)} \oplus \mathbb{Z}_{(2)}, \\ H_*(B) &= \mathbb{Z}^3_{(0)} \oplus \mathbb{Z}^3_{(1)}, \\ H_*(A \cap B) &= \mathbb{Z}^3_{(0)} \oplus \mathbb{Z}^6_{(1)} \oplus \mathbb{Z}^3_{(2)}. \end{aligned}$$

So, we obtain the LES, in reduced homology

$$0 \to H_3(X) \to \mathbb{Z}^3 \xrightarrow{f_2} \mathbb{Z} \to H_2(X) \to \mathbb{Z}^6 \xrightarrow{f_1} \mathbb{Z}^2 \oplus \mathbb{Z}^3 \to H_1(X) \to \mathbb{Z}^3 \xrightarrow{f_0} \mathbb{Z} \oplus \mathbb{Z}^3 \to H_0(X) \to 0.$$

We see $f_2 : H_2(A \cap B) \to H_2(A) \oplus H_2(B)$ is the map in which F_1, F_2, F_3, the generators of $H_2(A \cap B)$ each map to map to F, the generator of $H_2(A)$, as the boundary relation identifies each copy of $S^1 \times S^1$. That is, $f_2(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3$. Thus, $H_3(X) = \ker f_2 = \mathbb{Z}^2$. We also note f_2 is surjective, implying we have an epi-mono splitting giving us the LES

$$0 \to H_2(X) \to \mathbb{Z}^6 \xrightarrow{f_1} \mathbb{Z}^2 \oplus \mathbb{Z}^3 \to H_1(X) \to \mathbb{Z}^3 \xrightarrow{f_0} \mathbb{Z} \oplus \mathbb{Z}^3 \to H_0(X) \to 0.$$

We then see $f_0 : H_0(A \cap B) \to H_0(A) \oplus H_0(B)$ is the map in which A_1, A_2, A_3, the generators of $H_0(A \cap B)$ each map to A, the generator of $H_0(A)$ for the same reasoning as above, and to B_1, B_2, B_3 respectively, the generators of $H_0(B)$. That is, $f_0(\alpha_1, \alpha_2, \alpha_3) = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1, \alpha_2, \alpha_3)$. Thus, $H_0(X) = \operatorname{coker} f_0 = \mathbb{Z}$. We additionally f_0 is injective so we have another epi-mono splitting, giving us the LES

$$0 \to H_2(X) \to \mathbb{Z}^6 \xrightarrow{f_1} \mathbb{Z}^2 \oplus \mathbb{Z}^3 \to H_1(X) \to 0.$$

Finally, analyzing our map, by similar logic as above, we have

$$f_1(a_1, a_2, a_3, b_1, b_2, b_3) = (a_1 + a_2 + a_3, b_1 + b_2 + b_3, a_1, a_2, a_3).$$

Thus, $H_2(X) = \ker f_2 = \mathbb{Z}^2$ and $H_1(X) = \operatorname{coker} f_2 = \mathbb{Z}$. Putting this all together, we obtain

$$H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 1, \\ \mathbb{Z}^2 & k = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

■

Fall 2021, 10 Consider the following subsets of $\mathbb{R}^3$

$$\begin{aligned} Z &= \{(0,0,z) \mid z \in \mathbb{R}\} \\ C_1 &= \{(\cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\} \\ C_2 &= \{(2+\cos\theta, \sin\theta, 0) \mid \theta \in \mathbb{R}\} \end{aligned}$$

Prove there is no self-homeomorphism on $\mathbb{R}^3$ taking $Z \cup C_1$ to $Z \cup C_2$.

Let $X_1 = Z \cup C_1$ and $X_2 = Z \cup C_2$. X_1 is a line going "through" the unit circle and X_2 circle with a line "tangent" to it[5]. We show that no self-homeomorphism exists.

Say there exists a self-homeomorphism f. Then, f maps $\mathbb{R}^3\backslash X_1$ to $\mathbb{R}^3\backslash X_2$. This induces an isomorphism of fundamental groups.

However, $\mathbb{R}^3\backslash X_1$ deformation retracts onto a torus, so $\pi_1(\mathbb{R}^3\backslash X_1) = \mathbb{Z}^2$, and $\mathbb{R}^3\backslash X_2$ deformation retracts onto $S^2 \vee S^1$, and thus $\pi_1(\mathbb{R}^3\backslash X_2) = \pi_1(S^2 \vee S^1) = \pi_1(S^2) * \pi_1(S^1) = \mathbb{Z}$. These groups are clearly not isomorphic (they are free $\mathbb{Z}$-modules of differing rank) thus we have a contradiction. ■

15.4 Spring 2022

Spring 2022, 1 Let M be a closed (compact, without boundary) $2n$-dimensional manifold, and let ω be a closed 2-form on M which is non-degenerate, i.e., for any $p \in M$, the map $T_pM \to T_p^*M$, $X \to i_X\omega(p)$ is an isomorphism. Show that the de Rham cohomology groups $H^{2k}_{dR} \neq 0$ for $0 \leqslant k \leqslant n$.

It suffices to show $H^{2n}_{dR} \neq 0$, as $[\omega^n] = [\omega^k] \wedge [\omega^{n-k}]$ for any $0 \leqslant k \leqslant n$. We will show $H^{2n}_{dR} \neq 0$, by showing ω is a closed form that is not exact. Consider the map $T_pM \times T_pM \to \mathbb{R}$ where

$$X \times Y \mapsto \omega(p)(X,Y).$$

This is a bilinear form as $\omega(p)$ is a multilinear map. However as it is alternating, we have

$$\omega(p)(X,Y) = -\omega(p)(Y,X).$$

Finally, we see ω is non-degenerate because $Y \to \omega(p)(X,Y)$ is exactly the map $i_X\omega(p)$, which is an isomorphism by hypothesis. That is, for any $X \in T_pM$, $\exists Y \in T_pM$ such that $\omega(p)(X,Y)$ is nonzero, which proves $\omega(p)$ is non-degenerate.

So we have a non-degenerate bilinear skew-symmetric form. Thus, there exists a basis

$$X_1, \dots, X_n, Y_1, \dots, Y_n \in T_pM$$

such that

$$\omega(p)(X_i, Y_j) = \delta_{ij},$$

and

$$\omega(p)(X_i, X_j) = \omega(p)(Y_i, Y_j) = 0.$$

Thus,

$$\omega(p) = \sum_{i=1}^{n} X_i^* \wedge Y_i^*,$$

[5] Any TikZ enjoyers here?

implying

$$\omega^n(p) = n!\, X_1^* \wedge \cdots \wedge X_n^* \wedge Y_1^* \wedge \cdots \wedge Y_n^*.$$

Thus,

$$\omega^n(p)(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = n!$$

implying ω^n is nowhere vanishing as $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ is a basis for T_pM, implying M is orientable. As ω^n is nowhere vanishing, $\int_M \omega^n \neq 0$, and as M is closed, this implies ω^n is not exact, as desired. ∎

Spring 2022, 2 Let M be a closed n-dimensional manifold. Let ω be a closed k-form on $M, 1 \leqslant k \leqslant n$. Prove that for any $p \in M$ there is another closed k-form τ which vanishes on a neighborhood of p and such that $[\omega] = [\tau] \in H^k_{dR}(M)$.

Let $\omega \in \Omega^k(M)$ be given such that ω is closed. Let $p \in M$ be given. Find $B \subset M$ such that B is diffeomorphic to an open ball in $\mathbb{R}^n$, and some $U \subset \overline{U} \subset B$. Let ϕ be a bump function supported on B so that $\phi \equiv 1$ on $\overline{U}$.

Consider $i : B \hookrightarrow M$. We note that $i^*(\omega)$ is still closed as $di^*(\omega) = i^*(d\omega) = i^*(0) = 0$. Thus, as $H^k(B) \cong H^k(\mathbb{R}^n) = 0$, we have that $i^*(\omega)$ is exact. Therefore, let $\eta \in \Omega^k(B)$ be such that $d\eta = i^*(\omega)$. Then consider

$$\tau = \omega - d(\phi\eta).$$

Since $\phi \equiv 1$ on U, $\tau = \omega - d(\phi\eta) = \omega - d\eta = 0$, and, by construction, $[\omega] = [\tau]$. ∎

Spring 2022, 3 Let M be a closed n-dimensional manifold and let Ω be a volume form (i.e., a nonvanishing n-form) on M. Given a vector field X on M, its divergence $\operatorname{div}(X)$ is the smooth function on M defined by the identity:

$$\mathcal{L}_X(\Omega) = \operatorname{div}(X)\Omega$$

where $\mathcal{L}_X$ denotes the Lie derivative with respect to X.
(a) (5 pts) Prove that $\int_M \operatorname{div}(X)\Omega = 0$.
(b) (5 pts) Express $\operatorname{div}(X)$ in local coordinates.

(a) Using Cartan's Magic formula, and noting that $d\Omega = 0$ as Ω is a top form and $\partial M = \varnothing$ as M is closed, we obtain

$$\begin{aligned} \int_M \operatorname{div}(X)\Omega &= \int_M \mathcal{L}_X(\Omega) \\ &= \int_M di_X(\Omega) + \int_M i_X d\Omega \\ &= \int_{\partial M} i_X(\Omega) = 0. \end{aligned}$$

(b) For $p \in M$, consider the chart (U, x), where $p \in U$. Thus, locally we have $\Omega = f dx^1 \wedge \cdots \wedge dx^n$ for some $f \neq 0$, and $X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}$. Thus, via Cartan's magic formula and as Ω is a top form, we locally have

$$\operatorname{div}(X)\Omega = \mathcal{L}_X(\Omega) = di_X(\Omega) = di_X(f dx^1 \wedge \cdots \wedge dx^n).$$

We first notice that, as f is a 0-form,

$$i_X(fdx^1 \wedge \cdots \wedge dx^n) = fi_X(dx^1 \wedge \cdots \wedge dx^n) + i_X f \wedge (dx^1 \wedge \cdots \wedge dx^n) = fi_X(dx^1 \wedge \cdots \wedge dx^n).$$

However,

$$\begin{aligned} i_X(dx^1 \wedge \cdots \wedge dx^n) &= \sum_{i=1}^{n}(-1)^i dx^i(X) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n}(-1)^i X^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n. \end{aligned}$$

We then have

$$di_X(fdx^1 \wedge \cdots \wedge dx^n) = d(fi_X(dx^1 \wedge \cdots \wedge dx^n)) = df \wedge i_X(dx^1 \wedge \cdots \wedge dx^n) + fdi_X(dx^1 \wedge \cdots \wedge dx^n).$$

Calculating, we find

$$\begin{aligned} df \wedge i_X(dx^1 \wedge \cdots \wedge dx^n) &= \sum_{i=1}^{n}(-1)^i \frac{\partial f}{\partial x^i} X^i dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} X^i dx^1 \wedge \cdots \wedge dx^n, \end{aligned}$$

and

$$\begin{aligned} fdi_X(dx^1 \wedge \cdots \wedge dx^n) &= fd\left(\sum_{i=1}^{n}(-1)^i X^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n\right) \\ &= f\sum_{i=1}^{n}(-1)^i \frac{\partial X^i}{\partial x^i} dx^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n} f\frac{\partial X^i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Thus,

$$\begin{aligned} \operatorname{div}(X)\Omega = di_X(fdx^1 \wedge \cdots \wedge dx^n) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} X^i dx^1 \wedge \cdots \wedge dx^n + \sum_{i=1}^{n} f\frac{\partial X^i}{\partial x^i} dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n}\left[\frac{\partial f}{\partial x^i} X^i + f\frac{\partial X^i}{\partial x^i}\right] dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{i=1}^{n}\left[\frac{1}{f}\frac{\partial f}{\partial x^i} X^i + \frac{\partial X^i}{\partial x^i}\right] fdx^1 \wedge \cdots \wedge dx^n, \end{aligned}$$

implying

$$\operatorname{div}(X) = \sum_{i=1}^{n} \frac{1}{f}\frac{\partial f}{\partial x^i} X^i + \frac{\partial X^i}{\partial x^i}.$$

∎

Spring 2022, 4 Let ω be a smooth 1-form on a manifold M and let X and Y be smooth vector fields on M. Use the Cartan formula for Lie derivatives to derive the following formula:

$$d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]).$$

Cartan's magic formula states, for X a vector field,

$$\mathcal{L}_X = di_X + i_X d.$$

Thus, we have

$$(L_X\omega)(Y) = di_X\omega(Y) + i_X d\omega(Y) = d(\omega(X))(Y) + d\omega(X,Y) = Y(\omega(X)) + d\omega(X,Y),$$

as $\omega(X) : M \to \mathbb{R}$ is a linear functional. However, we also note that, via the product rule in which L_X satisfies,

$$(L_X\omega)(Y) = L_X(\omega(Y)) - \omega(L_X(Y)) = X(\omega(Y)) - \omega([X,Y]),$$

as, again, $\omega(Y) : M \to \mathbb{R}$ is a linear functional. Combining these equations implies our result. ∎

Spring 2022, 5 Let $N \subset \mathbb{R}^n - \{0\}$ be a compact submanifold of dimension m. Show that N is transverse to almost all k-dimensional linear subspaces in $\mathbb{R}^n$. Here "almost all" means the set of subspaces that are not transverse to N has measure zero.

Let $U \subset \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}}$ denote the open set of linearly independent vectors in $(\mathbb{R}^n)^k$. Consider the map $F : (\mathbb{R}^k \backslash \{0\}) \times U \to \mathbb{R}^n$ via

$$(\alpha^1, \dots, \alpha^k, v_1, \dots, v_k) \mapsto \sum_i \alpha^i v_i.$$

Note that we may restrict F to $\mathbb{R}^n \backslash \{0\}$ as $\sum_i \alpha^i v_i = 0$ if and only if $\alpha^i = 0$ for all i as this set of vectors are linearly independent. We claim that F is a submersion.

Note that given $(\alpha^1, \dots, \alpha^k, v_1, \dots, v_k)$, there exists some i such that $\alpha^i \neq 0$, thus for any $h \in \mathbb{R}^n$

$$\begin{aligned} dF_{(\alpha^1,\dots,\alpha^k,v_1,\dots,v_k)}(0,\dots,0,0,\dots,h,\dots,0) = \lim_{t\to 0} \frac{1}{t}(&F(\alpha^1,\dots,\alpha^k,v_1,\dots,h+t,\dots,v_k) \\ &- F(\alpha^1,\dots,\alpha^k,v_1,\dots,v_k)) = \alpha^i h. \end{aligned}$$

As F is a submersion, F is transverse to $M \subset \mathbb{R}^n \backslash \{0\}$. Thus, by Thom's transversality theorem, for almost all $v = (v_1, \dots, v_k) \in U$ the map $F_v : \mathbb{R}^k \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is transverse to M. This implies that the k-dimensional linear subspace spanned by the set v is transverse to M for almost all linearly independent sets of k vectors. As this is true for almost all linearly independent sets of k vectors, this is true for almost all k-dimensional linear subspaces, as there is a surjection $P : U \to Gr(k,n)$, where $Gr(k,n)$ denotes the space of k-dimensional linear subspaces of $\mathbb{R}^n$. ∎

Spring 2022, 6 Describe all the connected covering spaces of $\mathbb{RP}^2 \vee \mathbb{RP}^2$. Here $\vee$ is the wedge sum.

See here for another write-up.

First note that any connected (locally path-connected, semi-locally simply connected) space X admits a simply connected double cover $\tilde{X}$, then its only connected covering spaces are X and $\tilde{X}$ we can see this because $\tilde{X}$ is the universal cover of X, and since it is a double cover, (proposition 1.39 of Hatcher) $\pi_1(X)$ must have order 2 (so must be $\mathbb{Z}/2\mathbb{Z}$) so its only subgroups are the trivial group (corresponding to the universal cover) and the whole group (corresponding to the trivial cover $X \to X$).

Therefore, the covering spaces for $\mathbb{R}P^2$ are $\mathbb{R}P^2$ and S^2. In particular, in the covering of $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$, when we have a S^2, since it's a double cover, there are two connecting points that can be wedge summed with coverings of the other, while $\mathbb{R}P^2$ has only one of these connecting points. Thus, the covering spaces that we have are as follows: we can have a chain that begins and ends with $\mathbb{R}P^w$ (since we need to ensure that we our covering degree is the same), with S^2's in the middle, we can have an even number "bracelet" of S^2, we can have an infinite chain that starts with $\mathbb{R}P^2$ that infinitely chains S^2's, and we can also have an infinite chain of S^2's. ■

Spring 2022, 7 Let X be a CW complex consisting of one vertex p, 2 edges a and b, and two 2-cells f_1 and f_2, where the boundaries of a and b map to p, the boundary of f_1 is mapped to the loop ab^3 (that is first a and then b), and the boundary of f_2 is mapped to ba^3.

(a) Compute the fundamental group $\pi_1(X)$ of X. Is it a finite group?

(b) (5 pts) Compute the homology groups of X with integer coefficients.

(a) The presentation of the fundamental group is given by

$$\pi_1(X) = \langle a, b \mid ab^3, ba^3 \rangle$$

The relationship $ab^3 = 1$ implies that $a = b^{-3}$ so that $ba^3 = 1$ implies $b(b^{-3})^3 = 1$ implying $b^8 = 1$. Similarly by symmetry, $a^8 = 1$. Noting that $a = b^{-3} = b^5$, we may reduce any word

$$a^{k_1} b^{k_1'} \dots a^{k_n} b^{k_n'} = b^{5k_1} b^{k_1'} \dots b^{5k_n} b^{k_n'} = b^{5k_1 + k_1' + \dots + 5k_n + k_n'} = b^r$$

where $r \equiv 5k_1 + k_1' + \dots + 5k_n + k_n' \mod 8$. Thus

$$\pi_1(X) = \mathbb{Z}/8\mathbb{Z},$$

which is clearly a finite group.

(b) We know that $H_n(X) = 0$ for $n \geqslant 3$ since there are no 3 simplices and up. Consider the sequence of complexes

$$\dots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

we note that $\partial_1(a) = \partial_1(b) = p - p = 0$ and $\partial_2(f_1) = a + 3b$ and $\partial_2(f_2) = 3a + b$. We then analyze $\ker(\partial_2)$. Let $\partial_2(\alpha f_1 + \beta f_2) = 0$. Then,

$$0 = \alpha(a + 3b) + \beta(3a + b) = a(\alpha + 3\beta) + b(3\alpha + \beta).$$

However, solving this system of equations gives us $\alpha = \beta = 0$, thus ∂_2 is injective. Therefore,

$$H_0(X) = \frac{\ker(\partial_0)}{\text{im}\,(\partial_1)} = \frac{\mathbb{Z}}{0} = \mathbb{Z},$$

$$H_1(X) = (\pi_1(X))_{\text{ab}} = \mathbb{Z}/8\mathbb{Z},$$
$$H_2(X) = \frac{\ker(\partial_2)}{\text{im}\,(\partial_3)} = \frac{0}{0} = 0.$$

For all $k < 0$ and $k > 2$, we have $\partial_k \equiv 0$ implying $H_k(X) = 0$.

■

Spring 2022, 8 Let X be a topological space and let $Y = (X \times [0,1])/\sim$, where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ for all $x, x' \in X$. Compute the homology groups of Y in terms of those of X.

$Y = S(X)$, the suspension of X. Follow the argument as in 10.1 to get $\tilde{H}_n(Y) = \tilde{H}_{n-1}(X)$ for all n. ■

Spring 2022, 9 Let M be a compact odd-dimensional manifold with nonempty boundary ∂M. Show that the Euler characteristics of M and ∂M are related by $\chi(M) = \frac{1}{2}(\partial M)$.

See the proof of Theorem 4.2. ■

Spring 2022, 10 Let $A \in GL(n+1, \mathbb{C})$. It induces a smooth map

$$\phi_A : \mathbb{CP}^n \to \mathbb{CP}^n, \quad [(z_0, \dots, z_n)] \mapsto [A\,(z_0, \dots, z_n)]\,,$$

where $[(z_0, \dots, z_n)]$ is the usual equivalence class of $(z_0, \dots, z_n)$ in $\left(\mathbb{C}^{n+1} - \{0\}\right)/(z_0, \dots, z_n) \sim (\lambda z_0, \dots, \lambda z_n)$, where $\lambda \in \mathbb{C}^\times$. (You do not have to check the smoothness of ϕ_A.)
(a) (3 pts) Show that the fixed points of ϕ_A correspond to eigenvectors of A (up to multiplication by $\mathbb{C}^\times$).
(b) (3 pts) Show that ϕ_A is a Lefschetz map if the eigenvalues of A all have multiplicity 1.
(c) (4 pts) Compute the Euler characteristic of $\mathbb{CP}^n$ by calculating the Lefschetz number of some ϕ_A.

(a) Suppose that $\phi_A(x) = x$. Then $[Ax] = [x]$ or in other words, $Ax = \lambda x$, for some $\lambda \in \mathbb{C}$. However, this is precisely what it means for x to be an eigenvector of A.

Conversely, if $Ax = \lambda x$ then $[\phi_A(x)] = [Ax] = [\lambda x] = [x]$ so that x is a fixed point of ϕ_A

(b) A Lefschetz map is one such that the graph$(f) := \{x, f(x) : x \in X\}$ is transversal to $\Delta = \{(x,x) : x \in X\}$. This simply requires that $(d\phi_A)_x$ to have no fixed points for x an eigenvector. Let $\lambda_0, \dots, \lambda_n$ be our eigenvalues and $v_0, \dots, v_n$ our eigenbasis. Note $\lambda_i \neq 0$ for all i as A is invertible. Then, in local coordinates, where $z_i \neq 0$, we have

$$(d\phi_A)_x\left(\frac{z_0}{z_i}, \dots, \hat{z_i}, \dots, \frac{z_n}{z_i}\right) = \left(\frac{\lambda_0}{\lambda_i}\frac{z_0}{z_i}, \dots, \hat{z_i}, \dots, \frac{\lambda_n}{\lambda_i}\frac{z_n}{z_i}\right).$$

Noting that all of our eigenvalues are distinct, we have that $(d\phi_A)_x$ has no fixed points and thus is a Lefschetz map.

From part (a) we know that graph(f) and Δ intersect at the eigenvectors of A. At the point $x \in \text{graph}(f) \cap \Delta$ the transversality condition states

$$\text{graph}(df_x) + \Delta_x = T_x(\mathbb{C}P^n) \times T_x(\mathbb{C}P^n)$$

(c) We have $\chi(\mathbb{C}P^n) = L(\mathrm{id}) = L(\phi_I)$ where I is the identity matrix. Note additionally that Lefschetz numbers are invariant under homotopy. As $GL(n+1,\mathbb{C})$ is connected, we note that ϕ_I is homotopic to ϕ_A for any matrix A. Choosing $A = \mathrm{diag}(1,2,\ldots,n+1)$, we may in fact compute $L(\phi_A)$ as, from part (b), we know ϕ_A is a Lefschetz map. However,

$$L(\phi_A) = \sum_{x \text{ eigenvectors}} \mathrm{sign}(\det((d\phi_A - \mathrm{id})_x).$$

However using our analysis from part (b), we have

$$d(\phi_A - \mathrm{id})_x = \mathrm{diag}\left(\frac{1}{i} - 1, \frac{2}{i} - 1, \ldots, \hat{i}, \ldots, \frac{n+1}{i} - 1\right).$$

As this is a complex matrix, its determinant will be positive, thus

$$L(\phi_A) = \sum_{x \text{ eigenvectors}} 1 = n+1.$$

■

15.5 Fall 2022

Fall 2022, 1 The Grassmannian $\mathrm{Gr}(k,n)$ is the set of all k-dimensional subspaces of $\mathbb{R}^n$. Explicitly construct the structure of a smooth manifold on $\mathrm{Gr}(k,n)$ using atlases. What is its dimension?

There is a solution in Lee (Example 1.36) and Peterson's notes (Section 1.2) (and Wikipedia). ■

Fall 2022, 2 The orthogonal group $O(n)$ is the set of $n \times n$ matrices M satisfying $M^TM = \mathrm{Id}$. Construct the structure of a smooth manifold on $O(n)$ by viewing it as the preimage of a regular value of a smooth map $\mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)/2}$. Prove that its tangent bundle is trivializable.

Let $F : M_n(\mathbb{R}) \to \mathrm{Sym}(M_n(\mathbb{R}))$ where $M \mapsto M^TM$. We note this map is well defined as $(M^TM)^T = M^T(M^T)^T = M^TM$, so $F(M)$ is always symmetric. Additionally note that $O(n) = F^{-1}(I)$. We show I is a regular value.

Let $A \in F^{-1}(I)$ and $C \in \mathrm{Sym}(M_n(\mathbb{R}))$ be given. Note that A is invertible with $A^T = A^{-1}$. Let $B = \frac{1}{2}AC$. Then,

$$\begin{aligned} dF_A(B) &= \lim_{t\to 0} \frac{1}{t}\left(F(A+tB) - F(A)\right) = A^TB + B^TA \\ &= A^T\frac{1}{2}AC + \frac{1}{2}(AC)^T A = \frac{1}{2}A^TAC + \frac{1}{2}A^TAC^T = C, \end{aligned}$$

as $C = C^T$ since C is symmetric.

To show its tangent bundle is trivializable, we may instead simply prove all Lie groups are parallelizable. See Theorem 13.1. ■

Fall 2022, 3 Let M be a closed oriented smooth n-manifold. Prove that for every $k \in \mathbb{Z}$, there exists a smooth map $f : M \to S^n$ of degree k.

Throughout this problem, we use the algebraic topology definition of degree, in which $f_* : H_n(M) \to H_n(S^n)$ is multiplication by $d = \deg(f)$. We first find maps $S^n \to S^n$ with degree k for any k via suspensions. See Subsection 12.1.1 for a more detailed explanation[6].

We first find a degree k map $S^1 \to S^1$. However, using the above definition, we note that maps $f_k : S^1 \to S^1$ where $z \mapsto z^k$ has degree k.

We then prove $\deg(Sf) = \deg(f)$, for any map $f : X \to Y$. Through the standard Mayer-Vietoris argument for suspensions (see 10.1), and naturality of the exact sequence, we obtain the commutative diagram

$$\begin{tikzcd} \tilde{H}_{n+1}(S(X)) \arrow[r, "\cong"] \arrow[d, "Sf_*"] & \tilde{H}_n(X) \arrow[d, "f_*"] \\ \tilde{H}_{n+1}(S(Y)) \arrow[r, "\cong"] & \tilde{H}_n(Y) \end{tikzcd}$$

Thus, suspensions preserve degrees. As $S^{n-1}(S^1) = S^n$, taking suspensions of $z \mapsto z^k$ gives us maps $f_k : S^n \to S^n$ of degree k.

We now find a degree 1 map $M \to S^n$. This idea is taken from this MSE post. Let $B \subset M$ be an open set homeomorphic to an open ball in $\mathbb{R}^n$. Let $q : M \to M/(M - B)$ be the quotient map. Note that $M/(M - B) \cong S^n$. By naturality of the exact sequence, we have the commutative diagram:

$$\begin{tikzcd} \tilde{H}_n(M \arrow[r, "\cong"] \arrow[d, "q_*"] & \tilde{H}_n(M, M - B) \arrow[d, "\cong"] \\ \tilde{H}_n(M/(M - B)) \arrow[r, "\cong"] & \tilde{H}_n(M/(M - B), (M - B)/(M - B)) \end{tikzcd}$$

The top map is an isomorphism as M is orientable. The bottom map is an isomorphism as $(M - B)/(M - B)$ is a single point and thus has trivial reduced homology. The right map is an isomorphism by excision (see Hatcher Proposition 2.22). Thus q_* is an isomorphism and thus q has degree 1 as desired.

Let $f = f_k \circ q$. Then $\deg(f) = \deg(f_k)\deg(q) = k$. To obtain a smooth map, as degree is invariant under homotopy, we use the Whitney Approximation theorem to homotope f to a smooth map with the same degree. ∎

Fall 2022, 4 Let M be a smooth manifold and let $\omega \in \Omega^1(M)$ be a nowhere vanishing smooth 1-form. Prove that the following are equivalent.
(a) $\ker(\omega)$ is integrable.
(b) $\omega \wedge d\omega = 0$.
(c) There exists some $\alpha \in \Omega^1(M)$ such that $d\omega = \alpha \wedge \omega$.

We show (a) $\implies$ (c) $\implies$ (b) $\implies$ (a).

(a) $\implies$ (c). Let $\ker(\omega)$ be integrable. Then, for every $p \in M$, there exists some chart (U, x) such that, without loss of generality, $\ker(\omega) = \operatorname{span}\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{m-1}}\right)$, where $m = \dim M$. Thus, $\omega|_U = f_U dx^m$ where f_U is a nowhere vanishing function. Thus, $d\omega|_U = df_U \wedge dx^m$. Letting

[6] This subsection also includes a way of doing this more directly.

$\alpha|_U = \frac{df_U}{f_U}$ (which is possible as f_U is nowhere vanishing), we note that

$$d\omega|_U = df_U \wedge dx^m = \frac{df_U}{f_U} \wedge f_U dx^m = (\alpha \wedge \omega)|_U.$$

Considering $\{\phi_U\}$ a partition of unity subordinate to $\{U\}$, we define

$$\alpha = \sum_U \phi_U \alpha_U,$$

and thus, at every point p,

$$\alpha \wedge \omega = \left(\sum_U \phi_U \alpha_U\right) \wedge \omega = \sum_U \phi_U (\alpha_U \wedge \omega) = \sum_U \phi_U d\omega = d\omega.$$

(c) $\implies$ (b). Letting $d\omega = \alpha \wedge \omega$, we have

$$\omega \wedge d\omega = \omega \wedge (\alpha \wedge \omega) = -(\omega \wedge \omega) \wedge \alpha = 0$$

as $\omega \wedge \omega = 0$ since $\omega \wedge \omega = -\omega \wedge \omega$ as $\omega \in \Omega^1(M)$.

(b) $\implies$ (a). Suppose $\omega \wedge d\omega = 0$. We show that $\ker(\omega)$ is involutive, which is equivalent to integrable by Frobenius's theorem. Let $X, Y \in \ker(\omega)$ be given. We wish to show that $[X, Y] \in \ker(\omega)$. Note however that we have

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]).$$

If we can show $d\omega(X, Y) = 0$, then $\omega([X, Y]) = 0$ and thus $[X, Y] \in \ker(\omega)$.

Letting $Z \in \mathfrak{X}(M)$ such that $Z \notin \ker(\omega)$, there exists some $p \in M$ such that $\omega_p(Z) \neq 0$. Thus,

$$\begin{aligned} 0 = \omega \wedge d\omega_p(X, Y, Z) &= \omega_p(X) d\omega_p(Y, Z) + \omega_p(Y) d\omega_p(X, Z) + \omega_p(Z) d\omega_p(X, Y) \\ &= \omega_p(Z) d\omega_p(X, Y). \end{aligned}$$

Thus, $d\omega_p(X, Y) = 0$. However as we can find such a Z for every $p \in M$, we note that $d\omega(X, Y) = 0$.

■

Fall 2022, 5 Let M be a $2n$-dimensional manifold. A symplectic form on M is a smooth closed 2-form in $\Omega^2(M)$ so that $\omega \wedge \ldots \wedge \omega \in \Omega^{2n}(M)$ is a volume form. (That is, nowhere vanishing) Determine all pairs of positive integers (k, ℓ) so that $S^k \times S^\ell$ has a symplectic form.

We show the only pairs are $k = \ell = 1$ and $k = \ell = 2$.

Note that if $S^k \times S^\ell$ is a symplectic form, as $S^k \times S^\ell$ is closed, we require ω^n to not be exact. As $[\omega^n] = [\omega^k] \wedge [\omega^{n-k}]$, this implies we require all even De Rham cohomologies to be nontrivial. We also need $k + \ell$ to be even.

Suppose $k > 2$ and $\ell > 2$. Then, by Künneth's formula,

$$\begin{aligned} H^2(S^k \times S^\ell) &= \left(H^2(S^k) \otimes H^0(S^\ell)\right) \oplus \left(H^1(S^k) \otimes H^1(S^\ell)\right) \oplus \left(H^2(S^k) \otimes H^0(S^\ell)\right) \\ &= (0 \otimes \mathbb{Z}) \oplus (0 \otimes 0) \oplus (0 \otimes \mathbb{Z}) \end{aligned}$$

$$= 0$$

So, we do not have any symplectic manifolds in this case.

Suppose $k = 2$ and $\ell > 4$. Then, by Künneth's formula,

$$\begin{aligned} H^4(S^2 \times S^\ell) &= H^2(S^2) \otimes H^2(S^\ell) \oplus H^1(S^2) \otimes H^3(S^\ell) \oplus H^0(S^2) \otimes H^4(S^\ell) \\ &= \mathbb{Z} \otimes 0 \oplus 0 \otimes 0 \oplus \mathbb{Z} \otimes 0 = 0 \end{aligned}$$

So, we do not have any symplectic manifolds in this case.

The possible candidates are $S^2 \times S^2$, $S^1 \times S^1$, and $S^2 \times S^4$.

$S^1 \times S^1$ is symplectic as it is an orientable two dimensional manifold, as the product of orientable manifolds is orientable, so any volume form is our symplectic form.

We show $S^2 \times S^2$ is symplectic. Let η be a volume form on S^2, which exists as S^2 is orientable, and let $\pi_i : S^2 \times S^2 \to S^2$ be projection onto the ith coordinate. We have, via Künneth, $\pi_1^*\eta \wedge \pi_2^*\eta$ is a volume form on $S^2 \times S^2$. Take $\omega = \pi_1^*\eta + \pi_2^*\eta$. Note that $\pi_i^*\eta \wedge \pi_i^*\eta = \pi_i^*(\eta \wedge \eta) = 0$ as $\eta \wedge \eta$ is a 4-form on S^2. Thus,

$$\omega \wedge \omega = 2\pi_1^*\eta \wedge \pi_2^*\eta,$$

which is a volume form, as desired.

$S^2 \times S^4$ is not symplectic. Suppose it were. Note that, via Künneth,

$$H^2(S^2 \times S^4) = H^2(S^2) \otimes H^0(S^4) \cong \mathbb{Z}.$$

Thus, it is spanned by $\pi_1^*\eta$ where η is a volume form on S^2. Suppose ω were a symplectic form on $S^2 \times S^4$. Then, $[\omega] = c[\pi_1^*\eta]$. Thus, $[\omega^3] = c^3[\pi_1^*\eta^3] = 0$. However this contradicts the fact the ω^3 is a volume form, as it cannot be exact. ∎

Fall 2022, 6 Let C_* be a chain complex of free abelian groups. Let $A_* = C_* \otimes \mathbb{Z}/p$ and let $B_* = C_* \otimes \mathbb{Z}/p^2$ be the chain complexes we get by tensoring C_* degreewise with $\mathbb{Z}/p$ and $\mathbb{Z}/p^2$, respectively.

(a) Show that we have a short exact sequence of chain complexes

$$0 \to A_* \to B_* \to A_* \to 0$$

induced by the corresponding sequences of abelian groups

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0.$$

(b) Show how to define a Bockstein natural transformation

$$\beta : H_k(A_*) \to H_{k-1}(A_*)$$

such that we have an associated long exact sequence

$$\cdots \to H_k(A_*) \to H_k(B_*) \to H_k(A_*) \xrightarrow{\beta} H_{k-1}(A_*) \to \ldots$$

(c) Show that if x and y are elements such that $d(x) = py$, then

$$\beta(\bar{x}) = \bar{y},$$

where the bars indicate the reduction modulo p of the corresponding classes.

(d) Show conversely that given an element $\bar{x} \in H_k(A_*)$, if $\beta(\bar{x}) = 0$, then we can find elements $x, y \in C_*$ such that x reduces to $\bar{x}$ modulo p and $d(x) \equiv p^2 y$ modulo p^3.

(a) As C_* is a chain of free abelian groups, $C_* \otimes -$, degree-wise, is an exact functor. Thus, as

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

is a short exact sequence, so is

$$0 \to A_* \to B_* \to A_* \to 0.$$

(b) This is the snake lemma. Consider the following chain complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & (A_*)_k \arrow[r, "\partial"] \arrow[d, "f"] & (A_*)_{k-1} \arrow[r] \arrow[d, "f'"] & \cdots \\
{} \arrow[r] & (B_*)_k \arrow[r, "\partial"] \arrow[d, "g"] & (B_*)_{k-1} \arrow[r] \arrow[d, "g'"] & \cdots \\
{} \arrow[r] & (A_*)_k \arrow[r, "\partial"] \arrow[d] & (A_*)_{k-1} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

We desire to construct a map $\beta : H_k(A_*) \to H_{k-1}(A_*)$. Let $c \in (A_*)_k$ be such that $\partial c = 0$. By exactness of columns, we know there exists $b \in (B_*)_k$ such that $g(b) = c$. By commutativity of the bottom square, we know that $g'(\partial b) = \partial g(b) = \partial c = 0$, the $\partial b \in \ker g'$. By exactness of columns again, we know there exists $a \in (A_*)_{k-1}$ such that $f'(a) = \partial b$. Note that such an a is unique as f' is injective. We define $\beta(c) = a$, and we show this is well defined.

Let b, b' be such that $g(b) = c = g(b')$. Note that $g(b - b') = g(b) - g(b') = c - c = 0$, thus $b - b' \in \ker g$. So exactness gives us some $w \in (A_*)_k$ such that $f(w) = b - b'$. Arguing as above, there exists some $a, a' \in (A_*)_{k-1}$ such that $f'(a) = \partial b$ and $f'(a') = \partial b'$. By commutativity of the square, we have $f'(\partial w) = \partial(f(w)) = \partial(b - b') = \partial b - \partial b' = f'(a) - f'(a') = f'(a - a')$. As f' is injective, we have $\partial w = a - a'$. Then, $a - a' \in \operatorname{im}(\partial)$, and thus our map β is well-defined.

(c) We follow the construction as in part (b). Let $x \in (C_*)_k$ and $y \in (C_*)_{k-1}$ with $d(x) = y$. Consider $\overline{x} \in (A_*)_k$. We note that $\partial\overline{x} = \overline{py} = 0$. Then note that $\widehat{x} \in (B_*)_k$ has the property that $f(\widehat{x}) = \overline{x}$, where $\widehat{x}$ denotes reduction modulo p^2 and $\overline{x}$ denotes reduction modulo p. Then, we have $\partial(\widehat{x}) = \widehat{py}$. We finally note that $\overline{y}$ has the property that $f'(\overline{y}) = \widehat{py}$. Thus $\beta(\overline{x}) = \overline{y}$, as desired.

(d) This is equivalent to showing $d(x) \equiv 0$ modulo p^2. Suppose $\beta(\overline{x}) = 0$. Let $\widehat{x} \in (B_*)_k$ be such that $f(\widehat{x}) = \overline{x}$, and let $\widehat{y}$ be such that $\partial\widehat{x} = \widehat{y}$. As $f'(0) = \widehat{y}$, we must have that $\widehat{y} = 0$ as f' is injective. That is, $\partial\widehat{x} = 0$. However this is exactly what we want to show.

■

Fall 2022, 7 Let H be a union of n lines through the origin in $\mathbb{R}^3$. Compute $\pi_1\left(\mathbb{R}^3 - H\right)$.

We first notice that $\mathbb{R}^3 - H$ deformation retracts onto $S^2 - P$ where P is a collection of $2n$ points. However, this is homotopy equivalent to $\mathbb{R}^2 - P'$ where P' is a collection of $2n - 1$ points. We prove that $\pi_1(\mathbb{R}^2 - P_k) = \circledast_k \mathbb{Z}$, the free product of k copies of $\mathbb{Z}$, where P_k is a collection of k distinct points in $\mathbb{R}^2$ via induction and Van Kampen[7].

This is clearly the case when $k = 0$ as $\mathbb{R}^2$ is simply connected so $\pi_1(\mathbb{R}^2) = 0$. Similarly, when $k = 1$, we note that $\mathbb{R}^2 - P_1$ deformation retracts to S^1, so $\pi_1(\mathbb{R}^2 - P_1) = \mathbb{Z}$.

Let $P_k = \{y_1, \ldots, y_k\}$, where $y_1^1 \leqslant y_2^1 \leqslant \cdots \leqslant y_k^1$, where y_i^1 is the first coordinate of y_i. WLOG, as these points are all distinct, there exists some i such that $y_i^1 < y_{i+1}^1$, as if they are all equal, it must be true for the second coordinate. Let $\epsilon = y_{i+1}^1 - y_i^1$. Then, considering the two open subsets of $\mathbb{R}^2$, $H_1 = \{(x, y) \in \mathbb{R}^2 \mid x < y_i^1 + \frac{2\epsilon}{3}\}$ and $H_2 = \{(x, y) \in \mathbb{R}^2 \mid x > y_i^1 + \frac{\epsilon}{3}\}$, we have that $y_i^1 \in H_1$ and $y_{i+1}^1 \in H_2$ but $P \cap (H_1 \cap H_2) = \varnothing$. Thus, by Van Kampen, as $H_1 \cap H_2$ is simply connected,

$$\pi_1(\mathbb{R}^2 - P_k) = \pi_1(\mathbb{R}^2 - P_a) * \pi_1(\mathbb{R}^2 - P_b) = \circledast_a \mathbb{Z} * \circledast_b \mathbb{Z} = \circledast_k \mathbb{Z},$$

as $a + b = k$ and we obtained $\pi_1(\mathbb{R}^2 - P_a) = \circledast_a \mathbb{Z}$ and $\pi_1(\mathbb{R}^2 - P_b) = \circledast_b \mathbb{Z}$ by induction as $a, b < k$.

■

Fall 2022, 8 Let X be a path connected, locally path connected, semilocally path connected space. Recall that a path connected covering space $\tilde{X} \to X$ is abelian if $\pi_1(\tilde{X}$ is normal in $\pi_1(X)$ and the quotient is abelian. Show that there is an universal abelian cover: this is an abelian cover $\tilde{X} \to X$ such that for any other abelian cover $\tilde{Y} \to X$, there is a covering map $\tilde{X} \to \tilde{Y}$ factoring the map $\tilde{X} \to X$.

[7] You can also try simply stating that $\mathbb{R}^2 - P'$ deformation retracts to a wedge sum of $2n - 1$ S^1's, but I figure it is nice to add a formal argument which may be useful in other scenarios as well.

As X is a path connected, locally path connected, semilocally path connected space, it admits a universal cover $U \to X$. Let $\pi_1(X) = G$. Then, by the Galois correspondence of covering spaces, there exists a covering space $\tilde{X} \xrightarrow{p} X$ such that $p_*(\pi_1(\tilde{X}, \star)) = [G, G]$, the commutator subgroup of G. By definition, we have that $[G, G] \lhd G$ and $G/[G, G]$ is abelian. Thus $\tilde{X}$ is an abelian cover – we now show it is universal.

Indeed, we note that for any $N \lhd G$, G/N is abelian if and only if $[G, G] \subset N$. Thus, noting that if $Y \xrightarrow{q} X$ is an abelian cover, then $q_*(\pi_1(Y, \star)) = N$, and we have

$$p_*(\pi_1(\tilde{X}, \star)) = [G, G] \subset N = q_*(\pi_1(Y, \star)).$$

Thus, by the lifting property of covering maps, there exists some $\tilde{p} : \tilde{X} \to Y$ such that $p = q \circ \tilde{p}$.

It remains to show that $\tilde{p}$ is a covering map. To see this, let $y \in Y$ be given. We can choose a neighborhood $U \subset X$ of $q(y) = x$ such that $q^{-1}(U) \ni y$ is a disjoint union of open sets that are mapped homeomorphically to U. Then $\tilde{p}^{-1}(q^{-1}(U)) = p^{-1}(U)$ which is a disjoint union of open sets that are mapped homeomorphically by $q\tilde{p}$ to U. Equivalently, it is a disjoint union of open sets that are mapped homeomorphically by $\tilde{p}$ to $q^{-1}(U)$. Thus $\tilde{p}$ is a covering map onto Y. ■

Fall 2022, 9 The space $S^1 \times S^1$ is the mapping cone of the map

$$[a, b] : S^1 \to S^1 \vee S^1,$$

representing the commutator of the inclusion of the left summand $a : S^1 \to S^1 \vee S^1$ and the inclusion of the right summand $b : S^1 \to S^1 \vee S^1$. Use this and the long exact sequence to compute the homology.

We have that, by above,

$$S^1 \times S^1 = \left(S^1 \times [0, 1] \coprod (S^1 \vee S^1)\right) / \sim,$$

where $(x, 1) \sim [a, b](x)$ for all $x \in S^1$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in S^1$. Via the argument in 10.3, we obtain the LES

$$\cdots \to \widetilde{H}_n(S^1) \xrightarrow{[a,b]_*} \widetilde{H}_n(S^1 \vee S^1) \to \widetilde{H}_n(S^1 \times S^1) \to \dots$$

That is, plugging in the homologies of $H_n(S^1)$ and $H_n(S^1 \vee S^1)$, we get

$$0 \to \widetilde{H}_2(S^1 \times S^1) \to \mathbb{Z} \xrightarrow{[a,b]_*} \mathbb{Z}^2 \to \widetilde{H}_1(S^1 \times S^1) \to 0.$$

We first note that, on first homology, $[a, b]_*(x) = x + x - x - x = 0$, where x is the generator of $\widetilde{H}_1(S^1)$. This implies that $\mathbb{Z} \cong H_2(S^1 \times S^1)$ and $\mathbb{Z}^2 \cong H_1(S^1 \times S^1)$. We finally note that $H_0(S^1 \times S^1)$ as this space is path connected, via the definition of the mapping cone and S^1 and $S^1 \vee S^1$ are both path connected. ■

Fall 2022, 10 Let $f : X \to Y$ be a continuous, pointed map. Let $\Sigma^n(f) : \Sigma^n X \to \Sigma^n Y$ be the nth (pointed) suspension of f. Show that if for some $n, \Sigma^n(f)$ induces the trivial map on reduced homology, then it does for all n.

We prove that $\Sigma^{n-1}(f)$ and $\Sigma^{n+1}(f)$ induces the trivial map on reduced homology. Once this is proven, we may iterate $1 \leqslant k \leqslant n$ many times to obtain $\Sigma^{n-k}(f)$ induces the trivial map on reduced homology and for any k, we obtain $\Sigma^{n+k}(f)$ induces the trivial map on reduced homology. WLOG we may assume $n > 0$ as if not, we need only prove that $\Sigma^{n+1}(f)_*$ is trivial.

We first begin with a similar argument as in 10.1 to get, for any integer $m > 0$, $\widetilde{H}_{k+1}(S^m X) \cong \widetilde{H}_k(S^{m-1}X)$ for all k. This also gives us the commutative diagram, for all k,

$$\begin{tikzcd}
\tilde{H}_{k+1}(S^m(X)) \arrow[r, "\cong"] \arrow[d, "S^m(f)_*"] & \tilde{H}_k(S^{m-1}(X)) \arrow[d, "S^{m-1}(f)_*"] \\
\tilde{H}_{k+1}(S^m(X)) \arrow[r, "\cong"] & \tilde{H}_k(S^{m-1}(X))
\end{tikzcd}$$

Then, again for any integer $m > 0$, considering $\Sigma^m X = S^m X/(\{p\} \times [0,1])$, we note we have the LES, as $(S^m X, \{p\} \times [0,1])$ is a good pair,

$$\cdots \to \widetilde{H}_k(\{p\} \times [0,1]) \to \widetilde{H}_k(S^m X) \to \widetilde{H}_k(\Sigma^m X) \to \ldots$$

As $\{p\} \times [0,1]$ is contractible, this implies

$$\widetilde{H}_{k+1}(\Sigma^m X) \cong \widetilde{H}_{k+1}(S^m X) \cong \widetilde{H}_k(S^{m-1}X),$$

for all k. This, again, gives rise to the commutative diagram, for all k,

$$\begin{tikzcd}
\tilde{H}_k(S^m(X)) \arrow[r, "\cong"] \arrow[d, "\Sigma^m(f)_*"] & \tilde{H}_k(\Sigma^m(X)) \arrow[d, "S^m(f)_*"] \\
\tilde{H}_k(S^m(X)) \arrow[r, "\cong"] & \tilde{H}_k(\Sigma^m(X))
\end{tikzcd}$$

As $\Sigma^n(f)_*$ is trivial, so is $S^n(f)_*$, and thus so is $S^{n-1}(f)_*$ and $S^{n+1}(f)_*$, implying the same for $\Sigma^{n-1}(f)_*$ and $\Sigma^{n+1}(f)_*$. ∎

15.6 Spring 2023

Spring 2023, #1
Let M, N be smooth manifolds and $F : M \to N$ a smooth proper map.

(a) Show that F maps closed sets to closed sets.
(b) Show that the set of regular values is open.
(c) Let $C \subset N$ be compact. Show that for every open set $U \subset M$ containing $F^{-1}(C)$ there is an open set $V \subset N$ containing C, such that $F^{-1}(V) \subset U$.

(a) Since F is proper, the inverse image of a compact set $V \subset N$ is compact in M. Suppose A is closed, we desire to show
$$\overline{F(A)} \subset F(A).$$
Take a sequence $\{y_n\}_{n=1}^{\infty} \in F(A)$ such that $y_n \to y$. Since y_n belong in $F(A)$, it follows there exists a sequence x_n such that $F(x_n) = y_n$. Consider the compact set $K = \{y\} \bigcup \{y_n\}_{n=1}^{\infty}$. By sequential compactness, we know that there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$. However, $f(x_{n_k}) \to y$ so by uniqueness of limits $y = f(x)$. Therefore $y \in F(A)$ so $F(A)$ is closed.

(b) Let $A \subset M$ be the set of regular points of f. Let $x \in A$ and ϕ, ψ are coordinate charts around x and $f(x)$. Since x is a regular point, $d(\psi \circ f \circ \phi^{-1})$ is surjective at $\phi(x)$ (meaning there is a $n \times n$ minor with non-zero determinant). However, det is a continuous map, so $d(\psi \circ f \circ \phi^{-1})$ is also surjective at every point of some neighborhood of x. Therefore, this neighborhood is a subset of A, which means that A is open.

Now, let $B \subset N$ be the set of regular values. Then $y \in B$ iff $f^{-1}(y) \subset A$, which is the same as to say that $f(x) \neq A$ for every $x \in A^C$. In turn, this is equivalent to $y \notin f(A^C)$, so $B = (f(A^C))^C$. But since A is open, A^C is closed, so $f(A^C)$ is also closed, which means that $B = (f(A^C))^C$ is open.

(c) Notice that in order for $F^{-1}(V)$ to be contained in U, it needs to not intersect U^C, for which V needs to not intersect $f(U^C)$. So just take

$$V := (f(U^C))^C.$$

First, notice that since U is open and f is a closed map, V is open. So it is enough to show the two inclusions:

- Show that $V \supset C$ by supposing the contrary. Then $V^C \cap C \neq \varnothing$, so there exists $y \in V^C \cap C = f(U^C) \cap C$. Since $y \in f(U^C)$, there should be $x \in U^C$ s.t. $y = f(x)$. On the other hand, $y \in C$, so $x \in f^{-1}(C) \subset U$. Contradiction!
- Now, show that $F^{-1}(V) \subset U$ by supposing the contrary again. If this is not true, then $F^{-1}(V) \cap U^C \neq \varnothing$, so there should exist $x \in F^{-1}(V) \cap U^C$. Since $x \in F^{-1}(V)$, $f(x) \in V = (f(U^C))^C$. On the other hand, $x \in U^C$, so $f(x) \in f(U^C) = V^C$. Contradiction!

Thus the constructed V satisfies all the conditions.

■

Spring 2023, #2 Consider a smooth map $F : \mathbb{CP}^n \to \mathbb{CP}^n$.
(a) When n is even show that F has a fixed point.
(b) When n is odd give an example where F does not have a fixed point.

(a) If we can show $L(F) \neq 0$, we note that F has a fixed point by the Lefschetz fixed number theorem. Recall that the cohomology ring

$$H^*(\mathbb{C}P^{2k};\mathbb{Z}) = \frac{\mathbb{Z}[\alpha]}{\alpha^{2k+1}}, \quad |\alpha| = 2.$$

Let $F^*(\alpha) = m\alpha$, where $m \in \mathbb{Z}$ Then, by the cup product structure, we note that

$$F^*(\alpha^j) = m^j \alpha^j.$$

Thus,

$$L(F) = \sum_{j=0}^{k} (-1)^j Tr(F^* : H^j(\mathbb{C}P^2;\mathbb{Q}) \to H^j(\mathbb{C}P^2;\mathbb{Q})) = \sum_{j=0}^{2k} m^j.$$

If $m = 1$, then clearly $L(F) \neq 0$. Otherwise,

$$\sum_{j=0}^{2k} m^j = \frac{m^{2j+1} - 1}{m - 1} \neq 0.$$

(b) Consider $F : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}^{n+1}\backslash\{0\}$

$$(z_0, z_1, ..., z_n, z_{n+1}) \mapsto (-\overline{z_1}, -\overline{z_0}, ..., -\overline{z_{n+1}}, \overline{z_n}).$$

Note that
$$F(\lambda \mathbf{z}) = \overline{\lambda} F(\mathbf{z}).$$

Thus, considering $\pi : \mathbb{C}^{n-1}\backslash\{0\} \to \mathbb{C}P^n$, we have that $\pi \circ F$ factors through π, and thus we get a well-defined map $\tilde{F} : \mathbb{C}P^n \to \mathbb{C}P^n$ where

$$[z_0, z_1, ..., z_n, z_{n+1}] = [-\overline{z_1}, \overline{z_0}, ..., -\overline{z_{n+1}}, \overline{z_n}].$$

However, if this is the case, then $z_{2i} = -\lambda \overline{z_{2i-1}} = -\lambda\overline{\lambda} z_{2i} = -|\lambda|^2 z_{2i}$ and $z_{2i-1} = \lambda \overline{z_{2i}} = -\lambda\overline{\lambda} z_{2i-1} = -|\lambda|^2 z_{2i-1}$ which implies that $\lambda = 0$ or $\mathbf{z} = 0$, which is impossible.

■

Spring 2023, #3 Let
$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$
be a 2 -form defined on $\mathbb{R}^3 - \{0\}$ and $S^2 \subset \mathbb{R}^3$ the unit sphere.

(a) Compute $\int_{S^2} i^*\omega$, where $i : S \to \mathbb{R}^3$ is the inclusion.

(b) Compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ is defined by $j(x, y, z) = (2x, 3y, 5z)$.

(a) As we are on S^2, we note that $x^2 + y^2 + z^2 = 1$ for all $(x, y, z) \in S^2$. Thus, $i^*\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$. Thus, we have

$$\begin{aligned}
\int_{S^2} i^*\omega &= \int_{S^2} x dy \wedge dz + y dz \wedge dx + z dx \wedge dy \\
&= \int_{D^3} d(x dy \wedge dz + y dz \wedge dx + z dx \wedge dy), && \text{(by Stokes theorem)} \\
&= \int_{D^3} 3 dx \wedge dy \wedge dz = 4\pi.
\end{aligned}$$

(b) We note that $j(S^2)$ is an ellipse that contains S^2. Let E be the region of this ellipse with S^2 removed. We first show that $d\omega$ is in fact closed. However, $\omega = f\alpha$ where

$$\alpha = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy \quad \text{and} \quad f = (x^2 + y^2 + z^2)^{-3/2}.$$

Then, $d\omega = d(f\alpha) = df \wedge \alpha + f d\alpha$. Notice that $d\alpha = 3dV$ and

$$df = -(x^2 + y^2 + z^2)^{-5/2}(3x dx + 3y dy + 3z dz).$$

Thus,
$$d\omega = df \wedge \alpha + f d\alpha = -3f dV + 3f dV = 0.$$

Considering the inclusion $k : E \hookrightarrow \mathbb{R}^3\backslash\{0\}$, we have

$$0 = \int_E k^*(d\omega) = \int_E dk^*(\omega) = \int_{\partial E} \omega = \int_{S^2} j^*\omega - \int_{S^2} i^*\omega.$$

Therefore,

$$\int_{S^2} j^*\omega = \int_{S^2} i^*\omega = 4\pi.$$

■

Spring 2023, #4 Let M be a connected compact manifold with non-empty boundary ∂M. Show that M does not retract onto ∂M.

Fall 2013 #2. If such a retraction exists, then we can use the following LES in relative homology, where $(M, \partial M)$ is a good pair (by assumption):

$$0 \to H_n(\partial M) \to H_n(M) \to H_n(M/\partial M) \xrightarrow{\partial} H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M) \to \cdots$$

We note that $H_n(\partial M; \mathbb{Z}/2) = 0$ as ∂M is $(n-1)$-dimensional, and

$$H_n(M; \mathbb{Z}/2) = H^0(M, \partial M; \mathbb{Z}/2) = H^0(M/\partial M; \mathbb{Z}/2) = 0$$

by Lefschetz duality.

As a retraction $r : M \to \partial M$ exists, we have that $\mathrm{id}_* = (i \circ r)_* = r_* \circ i_*$, thus i_* is injective. However, Lefschetz duality also implies that

$$H_n(M, \partial M; \mathbb{Z}/2) = H^0(M; \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

as M is connected. As ∂ is injective, we obtain

$$\mathbb{Z}/2 = H_n(M, \partial M; \mathbb{Z}/2) = \mathrm{im}\,(\partial) = \ker(i_*) = 0,$$

which is a contradiction.

■

Spring 2023, #5 Let $M^m \subset \mathbb{R}^n$ be a closed connected submanifold of dimension m. (a) Show that $\mathbb{R}^n \backslash M^m$ is connected when $m \leqslant n-2$. (b) When $m = n-1$ show that $\mathbb{R}^n \backslash M^m$ is disconnected by showing that the mod 2 intersection number $I_2(f, M) = 0$ for all smooth maps $f : S^1 \to \mathbb{R}^n$.

(a) Take two points $x, y \in \mathbb{R}^n \backslash M^m$. Let $S = \{x, y\}$. Then consider path $f : [0,1] \to \{x, y\}$ such that $f(0) = a$ and $f(1) = y$. It's clear that $f|_S$ is vacuously transversal to M^m since its image is $\{x, y\}$ neither of which are in M. Furthermore, as S is closed, it follows from the transversal extension theorem that there is a g, homotopic to f that is transversal to M with $\partial g = \partial f$. We now claim that g doesn't intersect M^m. For if it does, then at the point $z \in g([0,1]) \cap M^m$ we have the relationship

$$\dim(T_z(f([0,1]) + \dim(M) = \dim(\mathbb{R}^n)$$

However, the LHS is at most $m + 1 \leqslant n - 1$ while the RHS is n so we must have $g([0,1]) \subset \mathbb{R}^n \backslash M^m$, implying we may write $g : [0,1] \to \mathbb{R}^n \backslash M^m$.

(b) First we prove the given claim about intersection numbers of maps $f : S^1 \to \mathbb{R}^n$. As $\mathbb{R}^n$ is simply connected, we may homotope f to some $g : S^1 \to \mathbb{R}^n$ such that $g(S^1) = p$ where $p \notin M$. As mod 2 intesection numbers are invariant under homotopy, we have $I_2(f, M) = I_2(g, M) = 0$.

Suppose that $\mathbb{R}^n \backslash M^m$ is connected. As M is a closed submanifold of dimension m, given some $m \in M$, there exists some open subset $m \in U \subset \mathbb{R}^n$ and a chart such that $U \cap M = \{(x_1, \ldots, x_m, 0)\}$. Then, there exists some $\epsilon > 0$ such that $(x_1, \ldots, x_m, \epsilon), (x_1, \ldots, x_m, -\epsilon) \in U$ for some choice of $x_1, \ldots, x_m$. Denote these points as p and q respectively. Considering the path $\gamma_0 : [0, 1] \to \mathbb{R}^n$ where

$$\gamma_0(t) = (x_1, \ldots, x_m, (1 - 2t)\epsilon),$$

we note that $I_2(\gamma_0, M) = 1$. However, as $\mathbb{R}^n \backslash M^m$ is connected, there exists another path γ_1 connecting p and q entirely contained in $\mathbb{R}^n \backslash M$. Thus, $I_2(\gamma_1, M) = 0$. However, then the loop $\gamma = \gamma_0 \circ \gamma_1$ has intersection $I_2(\gamma, M) = I_2(\gamma_0, M) + I_2(\gamma_1, M) = 1$, which is a contradiction, as desired.

■

Spring 2023, #6

(a) If X is a finite CW complex and $\tilde{X} \to X$ is a path-connected n-fold covering map, then show that the Euler characteristics are related by the formula

$$\chi(\tilde{X}) = n\chi(X).$$

(b) Let $X = \Sigma_g$ be a closed genus g surface. What path-connected, closed surfaces can cover X?

(a) Given an m-dimensional CW-complex X, one can lift the CW-structure to a CW-structure on $\tilde{X}$ by lifting the characteristic maps $\phi_a : D^k \to X$ to the cover $p : \tilde{X} \to X$, which can be done since $\pi_1(D^k) = 0$.

If the degree of p is n, there are exactly n lifts of ϕ_a to Y. So, for each k-cell e^k in X, there are n k-cells in the lifted CW-structure on $\tilde{X}$ which are mapped homeomorphically onto e^k. Thus,

$$\chi(\tilde{X}) = \sum_{i=0}^{m} (-1)^i \tilde{C}_i = \sum_{i=0}^{m} (-1)^i n C_i = n \sum_{i=0}^{m} (-1)^i C_i = n\chi(X),$$

where $\tilde{C}_i$ is the number of i-cells in $\tilde{X}$ and C_i is the number of i-cells in X.

(b) We may construct a CW complex for X with 1 Let $\tilde{X}$ be a path-connected, closed surface of say genus g'.

■

Spring 2023, #7 A group G is divisible if for all n, the map $g \mapsto g^n$ from G to itself is surjective. Show that if X is a path-connected CW-complex and if $\pi_1(X, x)$ is a divisible group, then the only path-connected finite cover of X is X itself. (Hint: This can be proven directly or by first showing that a divisible group has no finite index subgroups.)

We first prove that $\pi_1(X) = G$ has no finite index subgroups with index greater than 1. Let H be a proper subgroup of index $n > 1$, and let G/H be the set of left cosets of H. Then, G is partitioned by the left cosets of H, namely,

$$G = H \coprod g_1 H \coprod g_2 H \coprod \cdots \coprod g_{n-1} H,$$

where $g_1, \ldots, g_{n-1} \notin H$. Since G is divisible, we can pick $k \in G$ so that $k^{n!} = g_1$. Then $k^{n!} H = H$ such that $g_1 = k^{n!} \in H$, which is a contradiction.

Thus, $\pi_1(X)$ has no finite index subgroups with index greater than 1. Let $\tilde{X} \xrightarrow{p} X$ be a path-connected finite covering space of X. Then, $p_*(\pi_1(\tilde{X}, \star)) \subset \pi_1(X, \star)$ corresponds to a finite index subgroup of $\pi_1(X, \star)$. As the only finite index subgroup of $\pi_1(X, \star)$ is itself, we must have that $p_*(\pi_1(\tilde{X}, \star)) = \pi_1(X, \star)$, implying that $\tilde{X}$ is a 1-sheeted path-connected covering space of X, thus, $\tilde{X} \cong X$. ∎

Spring 2023, #8 Let M^n be an n-manifold, and consider a small disk D^n embedded in M^n. Show that the inclusion

$$\overline{M^n - D^n} \hookrightarrow M^n$$

induces an isomorphism on π_1 if $n \geqslant 3$ and a surjection if $n \geqslant 2$.

Let $M^n = \overline{M^n - D^n} \cup D^n$ and $S^{n-1} \cong \overline{M^n - D^n} \cap D^n$. Since S^{n-1} is connected, Van Kampen's theorem tells us $f : \pi(D^n) * \pi(\overline{M^n - D^n}) \to \pi_1(M^n)$ is surjective. However, note that since $\pi(\overline{D^n})$ is simply connected when $n \geqslant 2$, so $\pi(D^n)$ is the trivial group. Therefore, $i_* : \pi(\overline{M^n - D^n}) \to \pi_1(M^n)$ is surjective. To show we have an isomorphism, we just need to show that N which is the normal subgroup induced by all the cycles of the intersection is trivial. However, the intersection is S^{n-1} which has trivial fundamental group when $n \geqslant 3$. The desired result follows. ∎

Spring 2023, #9 Find, as a function of n and m, the homology groups

$$H_*\left(\mathbb{RP}^{n+m}, \mathbb{RP}^n; \mathbb{Z}\right)$$

Consider the LES in relative homology for the pair $(\mathbb{R}P^{n+m}, \mathbb{R}P^n)$.

We have $H_i(\mathbb{R}P^n) = 0$ for all $i > n$, and $H_i(\mathbb{R}P^n) \to H_i(\mathbb{R}P^{n+m})$ is an isomorphism for $i < m$. From this, it follows that

$$H_i(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \cong H_i(\mathbb{R}P^{n+m}) \quad \text{for } i > n+1$$

and $H_i(\mathbb{R}P^{n+m}, \mathbb{R}P^n) = 0$ for $i < n$. For $i = n, n+1$, we have the exact sequence

$$0 \to H_{n+1}(\mathbb{R}P^{n+m}) \to H_{n+1}(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \to H_n(\mathbb{R}P^n) \to H_n(\mathbb{R}P^{n+m}) \to H_n(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \to 0.$$

There are two cases.

Suppose n is even. Then, $H_n(\mathbb{R}P^n) = 0$, thus $H_i(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \cong H_i(\mathbb{R}P^{n+m})$ for $i = m, m+1$.

Suppose n is odd. Then, $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$ and $H_n(\mathbb{R}P^{n+m}) = \mathbb{Z}/2\mathbb{Z}$, and our sequence takes the form

$$0 \to H_{n+1}(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2 \to H_n(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \to 0$$

where ϕ is the map induced by inclusion. When we consider the inclusion $\mathbb{R}P^n \subset \mathbb{R}P^{n+m}$, the top n-cell of the subspace gets an $(n+1)$-cell attached to it in the larger space via a map of degree 2, and from the cellular chain complex you see that this n-cell that generates $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$ also generates $H_n(\mathbb{R}P^{n+m}) \cong \mathbb{Z}/2$. In other words ϕ is surjective, and therefore $H_n(\mathbb{R}P^{n+m}, \mathbb{R}P^n) = 0$ and $H_{n+1}(\mathbb{R}P^{n+m}, \mathbb{R}P^n) \cong \ker(\phi) \cong \mathbb{Z}$.

■

Spring 2023, #10 Consider the CW-complexes $A = S^n \vee S^n, X = S^n \times S^n$, and $B = S^n \times [0,1]/* \times [0,1]$, where $*$ is the basepoint of S^n. There are inclusions $A \hookrightarrow X$ given by the pairs of points where at least one is the basepoint and $A \hookrightarrow B$ which takes one S^n to $S^n \times 0$ and the other to $S^n \times 1$. Compute the homology of

$$Y = X \cup_A B$$

Consider the 'thickening' of X and B and call it U and V respectively. Then $U \cap V$ deformation retracts to A. We obtain the LES

$$\cdots \to H_m(A) \to H_m(X) \oplus H_m(B) \to H_m(Y) \to H_{m-1}(A) \to \cdots$$

Via Van Kampen we obtain

$$\widetilde{H}_*(A) = \widetilde{H}_*(S^n) \oplus \widetilde{H}_*(S^n) = \mathbb{Z}^2_{(n)}.$$

Via Künneth we obtain

$$H_*(X) = H_*(S^n) \otimes H_*(S^n) = \mathbb{Z}_{(2n)} \oplus \mathbb{Z}^2_{(n)} \oplus \mathbb{Z}_{(0)}.$$

Finally, noting that B deformation retracts onto S^n, we have[8] $H_*(B) = H_*(S^n)$.

Firstly noting that X and B are path connected and "glued" together, our space Y is path connected. Thus $H_0(Y) \cong \mathbb{Z}$.

Case 1: $n = 1$. In this case, we obtain the sequence

$$0 \to \widetilde{H}_2(X) \to \widetilde{H}_2(Y) \to \widetilde{H}_1(A) \to \widetilde{H}_1(X) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(Y) \to 0.$$

Considering the map $f : \widetilde{H}_1(A) \to \widetilde{H}_1(X) \oplus H_1(B)$ induced by the relations on the boundary, we see that $(a,b) \mapsto (a,b,a+b)$. This map is injective, so we have an epi-mono split at $\widetilde{H}_2(Y) \to \widetilde{H}_1(A)$, implying $H_2(Y) \cong H_2(X) = \mathbb{Z}$. Additionally, we note that $H_1(Y) \cong \operatorname{coker} f = \mathbb{Z}\langle a,b,c\rangle / \mathbb{Z}\langle a,b,a+b\rangle \cong \mathbb{Z}$.

Case 2: $n > 1$.

In this case we must consider two portions of our LES. Namely

$$0 \to H_{2n}(X) \to H_{2n}(Y) \to 0$$

and

$$0 \to H_{n+1}(Y) \to H_n(A) \to H_n(X) \oplus H_n(B) \to H_n(Y) \to 0.$$

[8]This can also be derived via relative homology.

The first portion gives us that $H_{2n}(Y) \cong H_{2n}(X) \cong \mathbb{Z}$. In the second portion, we must analyze the map, however it is the same map as above; $f : H_n(A) \to H_n(X) \oplus H_n(B)$ is the map in which $(a, b) \mapsto (a, b, a + b)$. As this map is injective, we obtain $H_{n+1}(Y) = 0$ and $H_n(Y) \cong \operatorname{coker} f \cong \mathbb{Z}$.

Thus, we have

$$H_k(Y) = \begin{cases} \mathbb{Z} & k = 0, n, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

■