Affine Deodhar Diagrams and Rational Dyck Paths

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(k,n)-Deodhar Diagrams

A (k,n)-Deodhar diagram (shortened to **Deogram**) is a filling of boxes of a $k \times (n-k)$ rectangle with crossings, ┼, and elbows, ⌟⌜, with

1. Strand permutation equal to identity,
2. Exactly $n-1$ elbows,
3. No elbows after an odd number of crossings.

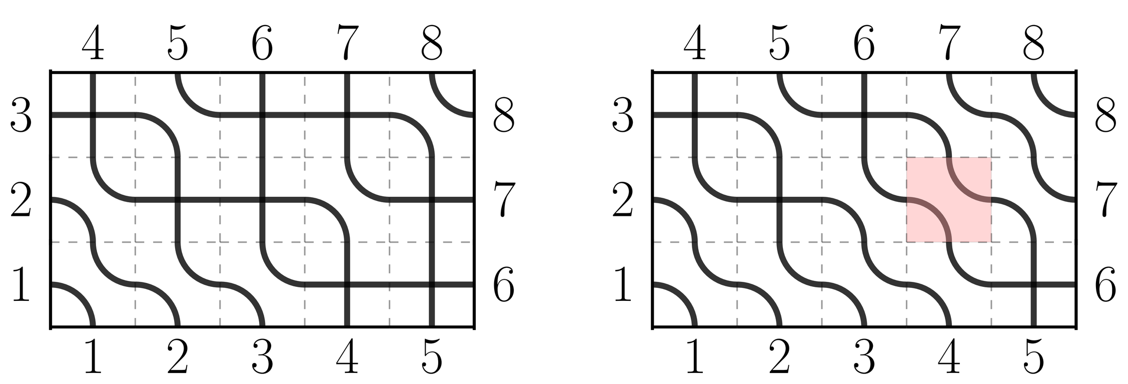
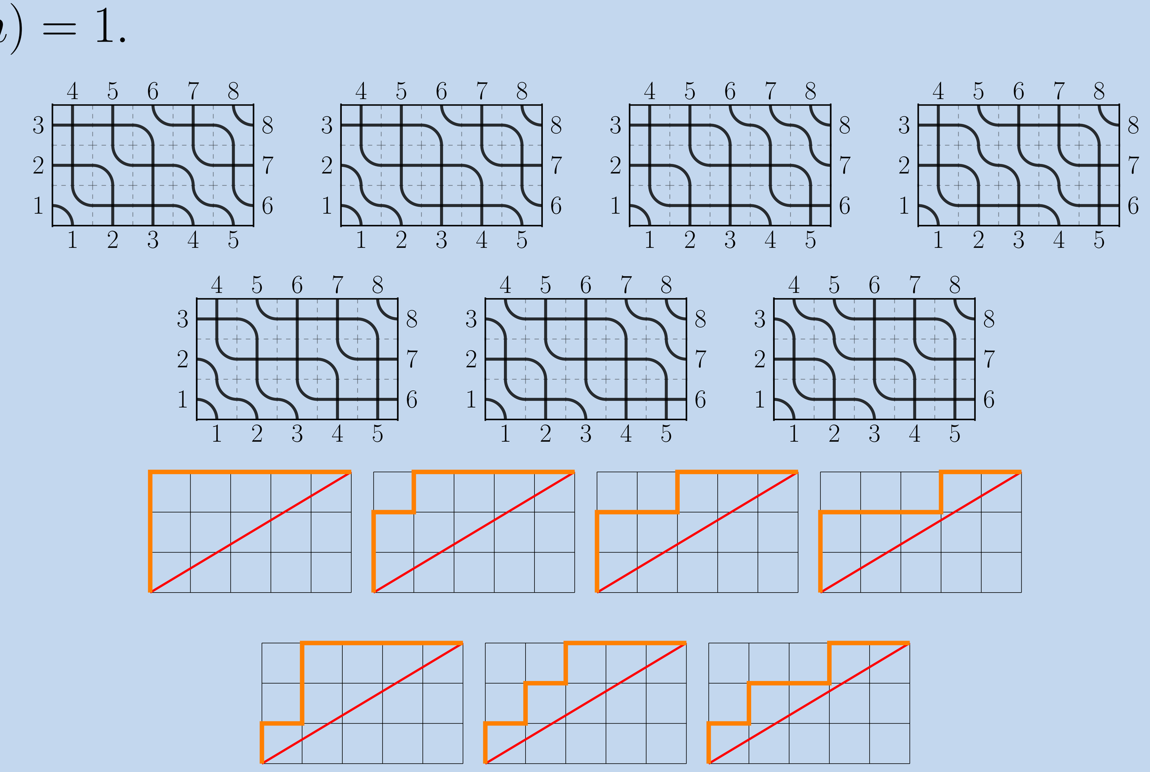


Figure 1. The left is an example of a $(3,8)$-Deogram. The filling on the right has too many elbows and violates condition 3 at the highlighted square.

Let $\mathrm{Deo}_{k,n}$ denote the set of (k,n)-Deograms.

Motivating Question

Theorem ([GL24]) $\#\mathrm{Deo}_{k,n} = \#\mathrm{Dyck}_{k,n}$ for $0 < k < n$ with $\gcd(k,n) = 1$.



Question: Can we find a bijection?

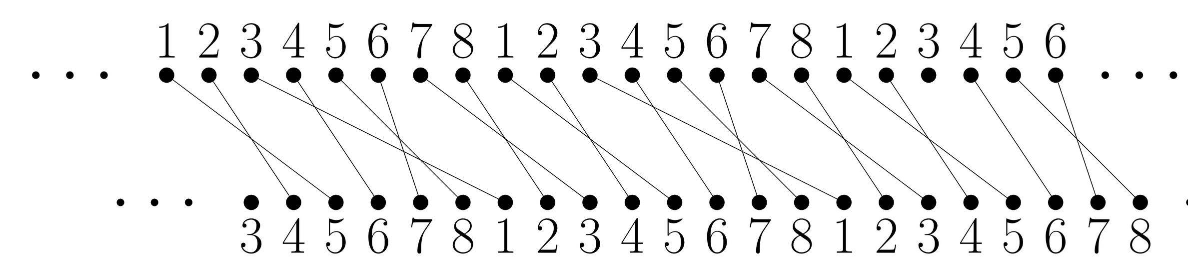
Main Result

Theorem (M., in preparation) For $0 < k < n$ with $\gcd(k,n) = 1$, we find a bijection

$$\mathrm{Deo}_{k,n} \to \mathrm{Dyck}_{k,n}.$$

Bounded Affine Permutations

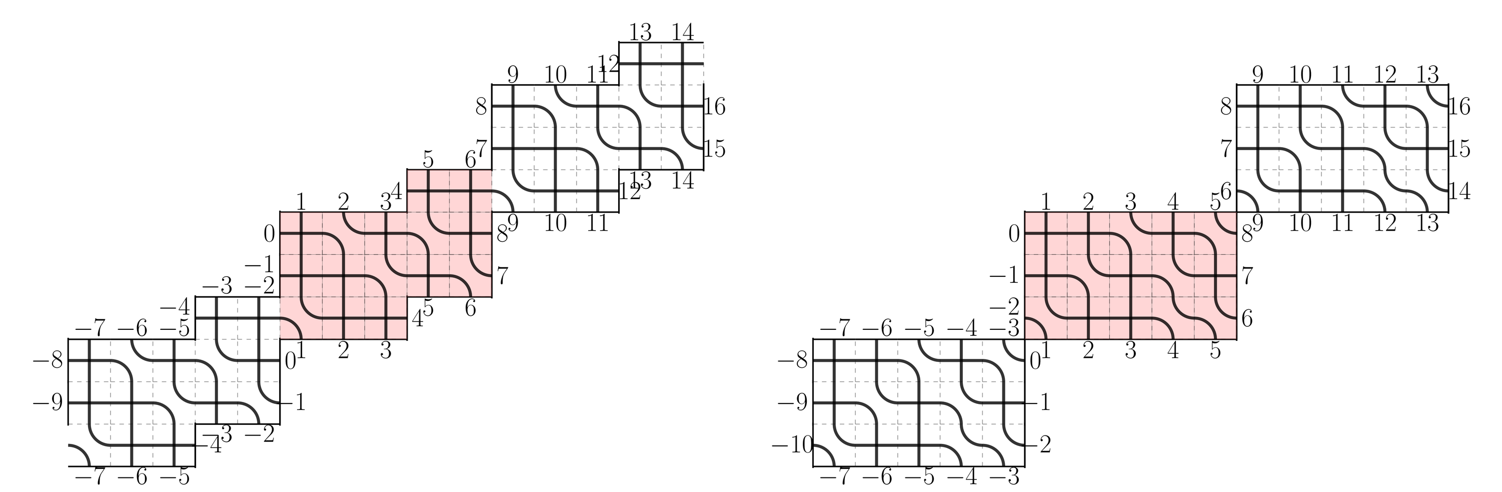
Let $\mathbf{B}_{k,n}$ denote the set of (k,n)-bounded affine permutations.



Main Tool: Affine Deograms

An f-**affine Deogram** is a filling of the space between a path P with k up-steps and $n-k$ right steps and its translate by k up-steps with:

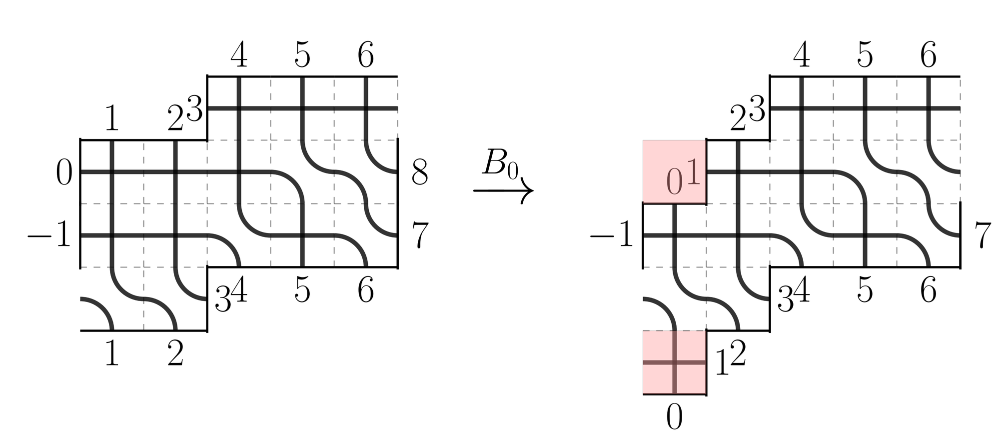
1. Strand permutation equal to f,
2. Exactly $n - (\#\text{cycles of } f)$ elbows (inside a red region),
3. No elbows after an odd number of crossings.



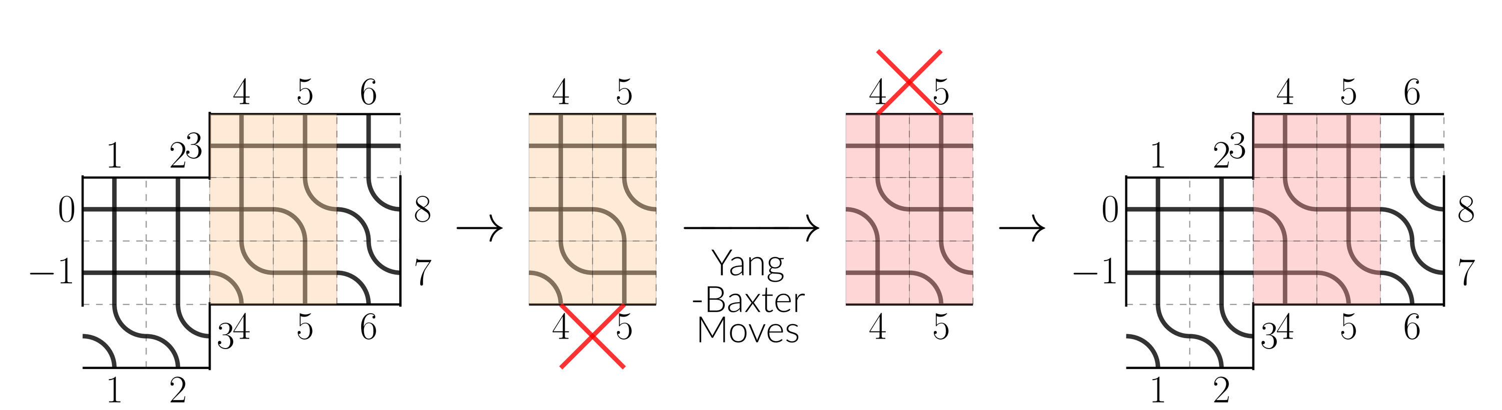
We let $\mathrm{AffDeo}_{f,P}$ denote the set of f-affine Deograms under P.

Our Moves on Affine Deograms

Box Addition/Removal. We change our path and move the box up/down.



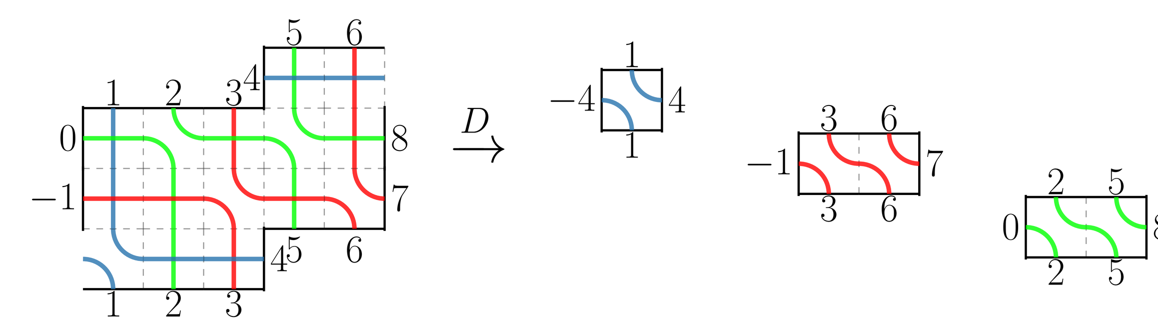
Zipper. We cross wires below and locally apply Yang-Baxter moves until the crossing moves to the top of the path.



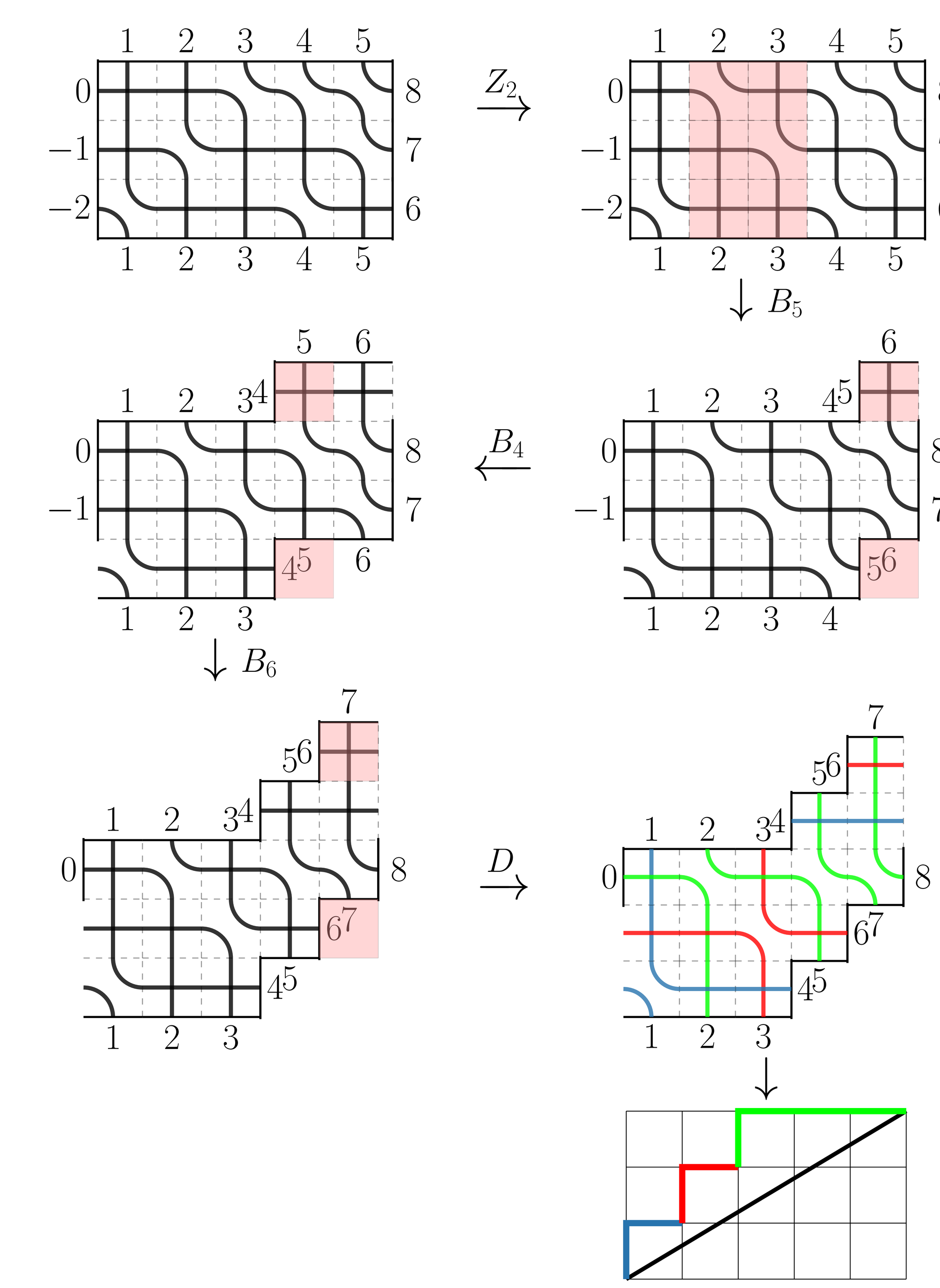
Decoupling and its Visualization

Let $f = f_1 f_2 \ldots f_r$ be a decomposition of $f \in \mathbf{B}_{k,n}$ into cycles. Then,

$$\#\,\mathrm{AffDeo}_{f,P} = \prod_{i=1}^{r} \#\,\mathrm{AffDeo}_{f_i,P_i}.$$



Full Recurrence Example



References

[GL24] Pavel Galashin and Thomas Lam. Positroid catalan numbers. *Communications of the American Mathematical Society*, 4(08):357–386, 2024.