Affine Deodhar Diagrams and Rational Dyck Paths

UCLA Combinatorics Forum

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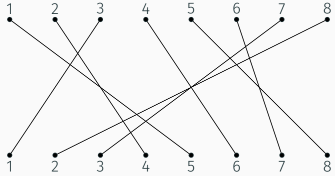
Permutations

A permutation $\bar{f} \in S_n$ is a bijection $\bar{f} : [n] \to [n]$, where $[n] = \{1, 2, \ldots, n\}$.

Let $k(\bar{f}) = \#\{i \,|\, \bar{f}(i) < i\}$.



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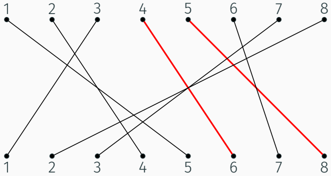

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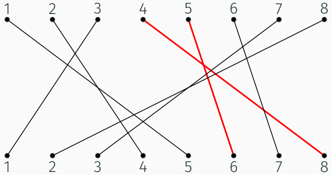

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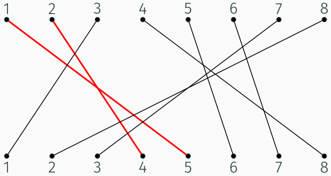

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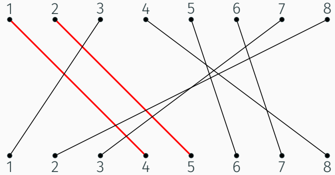

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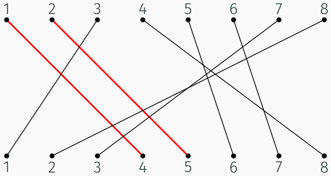
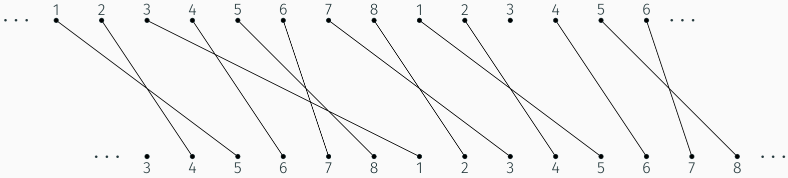


But I want the transposition of 1 and n to be simple..

Bounded Affine Permutations

For $\bar{f} \in S_n$, we can associate a **bounded affine permutation** $f : \mathbb{Z} \to \mathbb{Z}$ to $\bar{f}$ such that

1. $f(i) \equiv \bar{f}(i) \pmod{n}$ for $1 \leq i \leq n$,
2. $\sum_{i=1}^{n} f(i) - i = kn$,
3. $i \leq f(i) < i + n$ for all $i \in \mathbb{Z}$,
4. $f(i+n) = f(i) + n$ for all $i \in \mathbb{Z}$.



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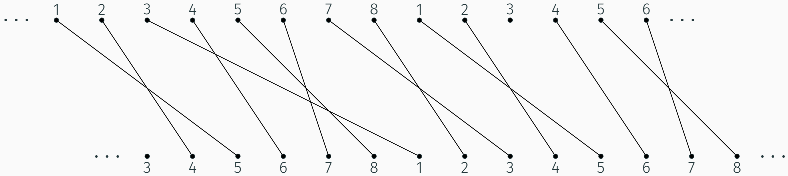

Let $\mathsf{B}_{k,n}$ denote the set of (k, n)-bounded affine permutations.

Overview

$$f \in \mathsf{B}_{k,n}$$

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Inversion set $\Gamma(f)$ $\leftarrow$ f $\rightarrow$ $(v, w) \in S_n \times S_n$

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$$
\begin{array}{ccc}
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\swarrow & & \searrow \\
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\mathrm{Dyck}(\Gamma(f)) & & \mathrm{Deo}_f^{\max}
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Inversion set $\Gamma(f)$ → Dyck$(\Gamma(f))$

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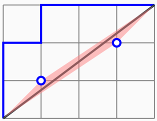

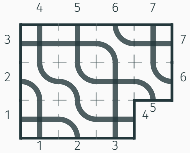

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Inversion set $\Gamma(f)$ → $\text{Dyck}(\Gamma(f))$

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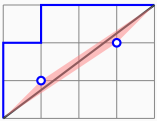
$\text{Dyck}(\Gamma(f)) \overset{?}{\longleftrightarrow} \text{Deo}_f^{\max}$



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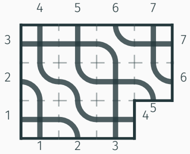
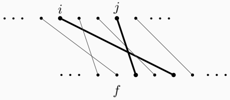
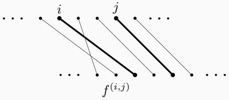


Inversion multiset $\Gamma(f)$

Resolving crossings.





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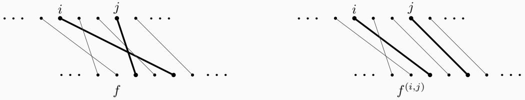
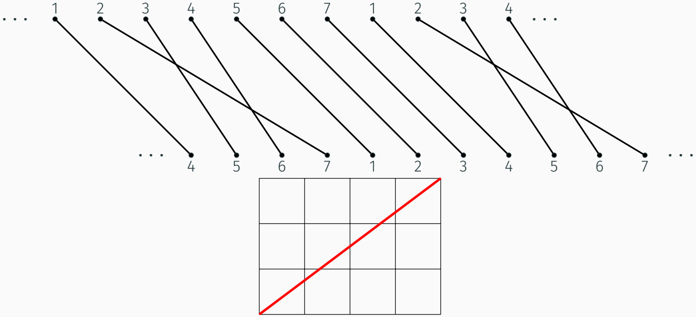


Inversion Multiset
The multiset $\Gamma(f)$ contains a point $\gamma(f_1^{(i,j)}) = (k, n-k)$ for each inversion $(i,j), i < j$, where f_1 is the cycle with i after resolving.

Inversion Multiset Example

Here, $f = [4, 7, 5, 6, 8, 9, 10]$, $k(f) = 3$, $n(f) = 7$.



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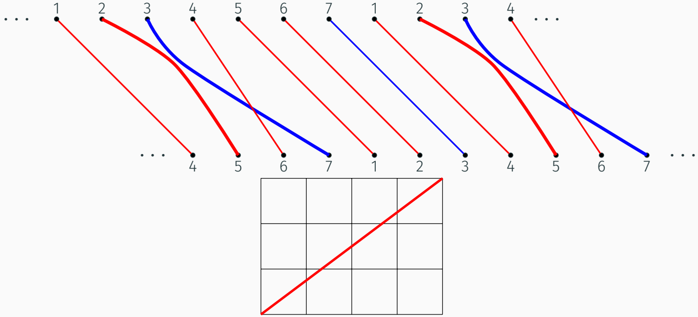

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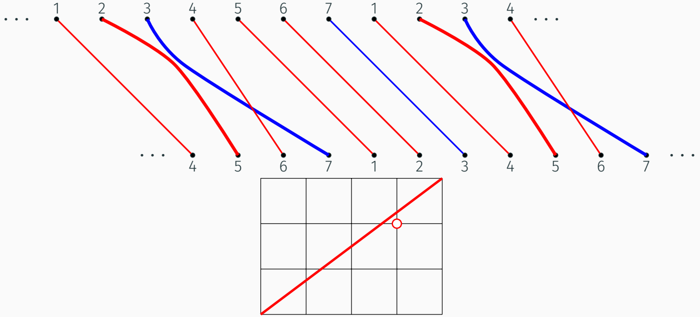

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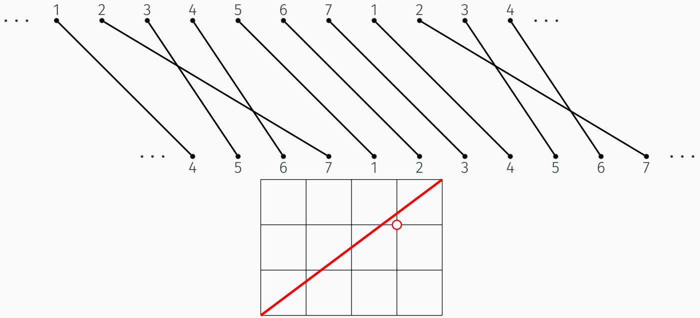

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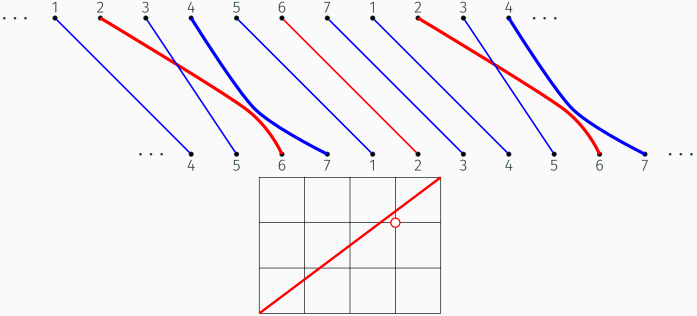

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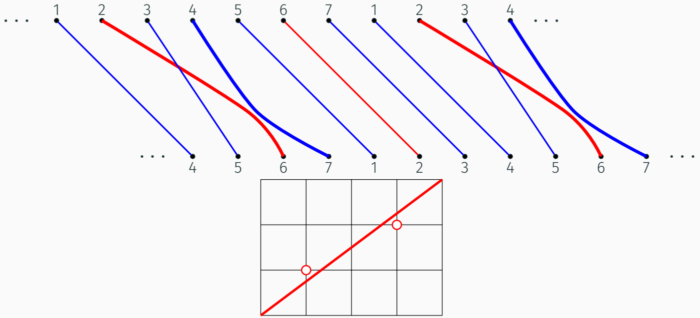

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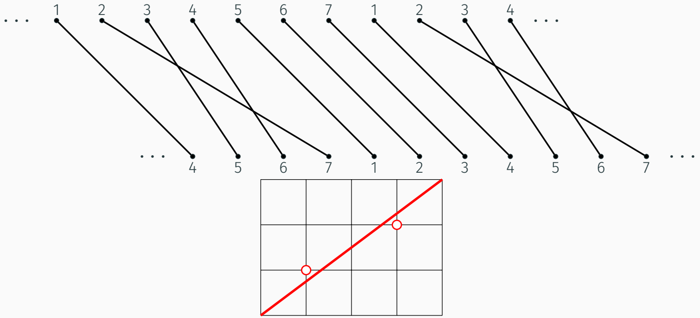

$\Gamma(f) = \{(1, 1), (2, 3)\}$

Properties of $\Gamma(f)$

Repetition-Free
When the multiset $\Gamma(f)$ is a set, we call f *repetition-free*. When $\Gamma(f)$ contains every lattice points of its convex hull, we call the set $\Gamma(f)$ convex.

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The set $\Gamma(f)$ is centrally symmetric and convex. For any centrally symmetric and convex region Γ, there exists a repetition-free f such that $\Gamma(f) = \Gamma$.

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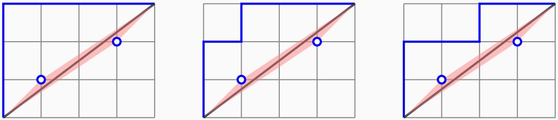
Some Generalized Catalan Number

Define $C_f := \#\,\mathrm{Dyck}(\Gamma(f))$.



For $f_{k,n}(i) = i + k$, $\Gamma(f) = \emptyset$, so $C_{f_{k,n}} = \#\,\mathrm{Dyck}_{k,n-k} = C_{k,n-k}$.

Definitions

Definition
For $w \in S_n$, we say w is k-Grassmannian if $w(i) > w(i+1) \Leftrightarrow i = k$.

Example: $w = (2, 4, 5, 8, 1, 3, 6, 7)$ is 4-Grassmannian.

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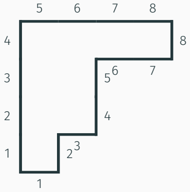
We have a bijection

$$\{w \in S_n \,|\, w\ k - \text{Grassmannian}\} \leftrightarrow \{\lambda \subseteq k \times (n-k) \text{ rectangle}\}$$



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Theorem (Knutson-Lam-Speyer, '13)
For bounded affine permutations f, we have a bijection

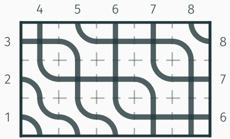
$$\{f \,|\, k(f) = k, n(f) = n\}$$
$$\updownarrow$$
$$\{(v,w) \in S_n \times S_n \,|\, w \text{ is } k - \text{Grassmannian and } v \leq w\}$$

Deograms

Deograms

A (maximal) f-*Deodhar diagram* (Deogram) for f, is a filling of a Young tableau of $\lambda(w)$ with crossings, , and elbows, , such that

1. The resulting strand permutation is v.
2. **Distinguished.** No elbows after an odd number of crossings (from top-left).
3. **Maximal.** Contains exactly $n - c(f)$ many elbows, where $c(f) = \#\text{cycles of } f$.



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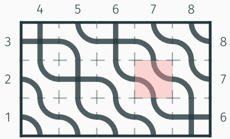

Non-example

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Example

$$\mathrm{RowSpan}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \in \mathrm{Gr}(2,3).$$

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For any $f \in \mathsf{B}_{k,n}$, we define a **positroid variety** $\Pi_f^\circ \subseteq \mathrm{Gr}_{\geq 0}(k, n)$.

Theorem (Knutson-Lam-Speyer, 2013)
We have a stratification

$$\mathrm{Gr}(k, n) = \bigsqcup_{f \in \mathsf{B}_{k,n}} \Pi_f^\circ.$$

Theorem (Deodhar, 1985)
For any field $\mathbb{F}$, we have a decomposition

$$\Pi_f^\circ = \bigsqcup_{D \in \mathrm{Deo}_f} (\mathbb{F}^*)^{\#\mathrm{elbows}(D)} \times \mathbb{F}^{(\#\mathrm{crossings}(D) - \ell(f))/2}.$$

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$$\#\Pi_f^\circ(\mathbb{F}_q) = \sum_{D \in \mathrm{Deo}_f} (q-1)^{\#\mathrm{elbows}(D)} q^{(\#\mathrm{crossings}(D) - \ell(f))/2}.$$

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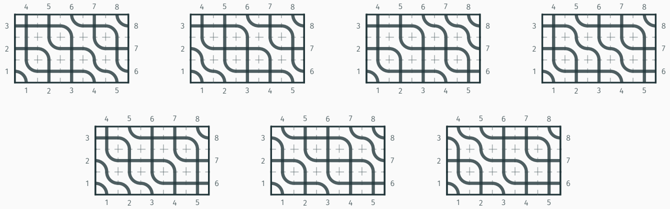
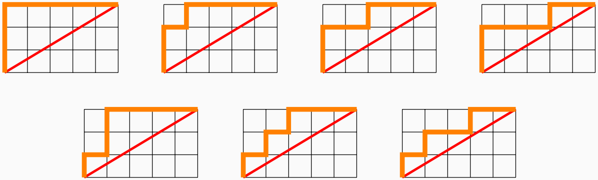
Some Positroid Numbers
Define $C_f = \# \mathrm{Deo}_f^{\max}$.

Positroid Catalan Numbers

Theorem (Galashin-Lam, '21)
For $0 < k < n$ with $\gcd(k, n) = 1$ and $f \in \mathrm{B}_{k,n}$ repetition-free,
$\# \operatorname{Deo}_f^{\max} = \# \operatorname{Dyck}(\Gamma(f))$.





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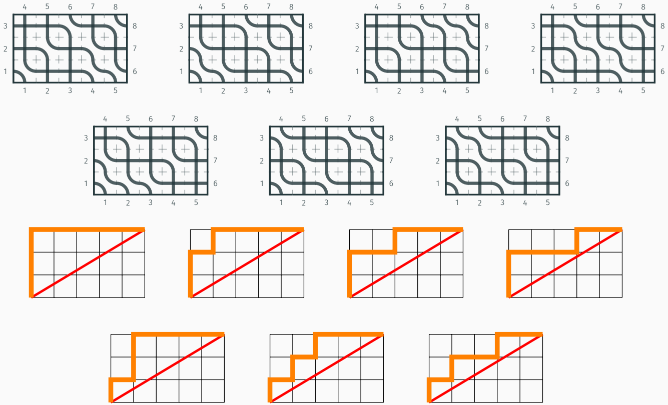

Definition
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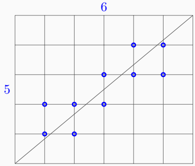
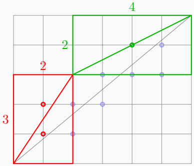
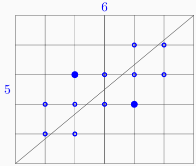
Theorem (M., '25+)
For $0 < k < n$ with $\gcd(k, n) = 1$ and $f \in \mathsf{B}_{k,n}$ repetition-free, we find a bijection $\mathsf{Deo}_f^{\max} \to \mathsf{Dyck}(\Gamma(f))$.

Dyck Path Recurrence

Let f_1, f_2 the cycles obtained by resolving f at $i, i+1$, and $f' = s_i f s_i$.







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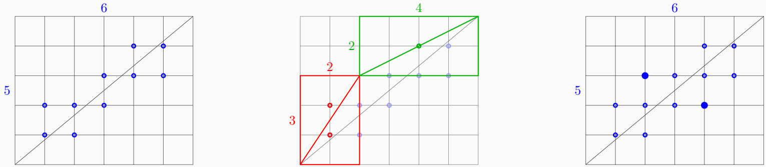

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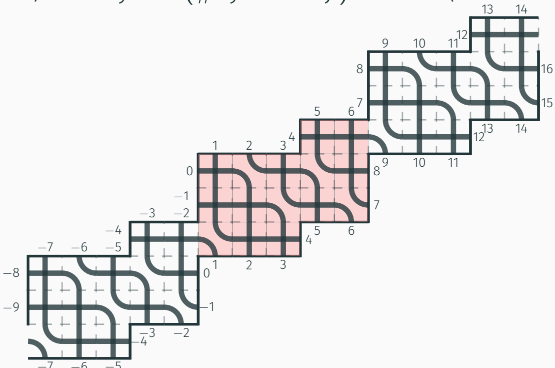
Goal: Find the same recurrence for Deograms.

Main Tool: Affine Deograms

A (maximal) f-**affine Deogram** is a *periodic filling* of the space between a path P with k up-steps and $n-k$ right steps and its vertical translate with:

1. Strand permutation equal to $f \in \mathsf{B}_{k,n}$,
2. (Distinguished) No elbows after an odd number of crossings,
3. (Maximal) Exactly $n-$ (#cycles of f) elbows (inside a red region).



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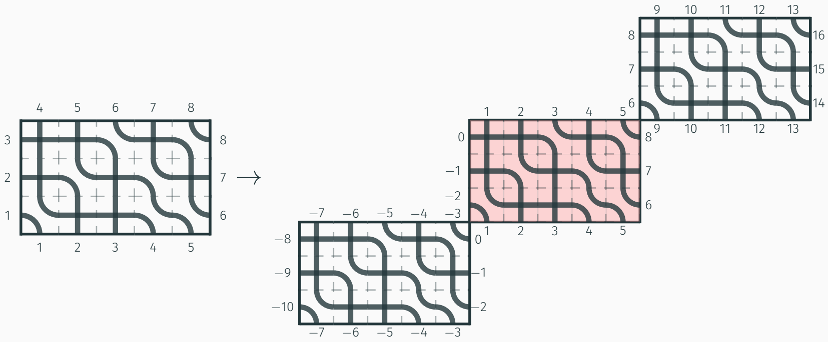
For some paths P, we have a bijection $\mathsf{Deo}_f \to \mathsf{AffDeo}_{f,P}$.



Moves on Affine Deograms

We have 3 moves on f-affine Deograms:

1. Box Addition/Removal
2. Zipper
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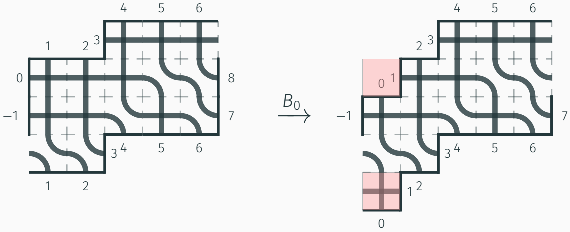

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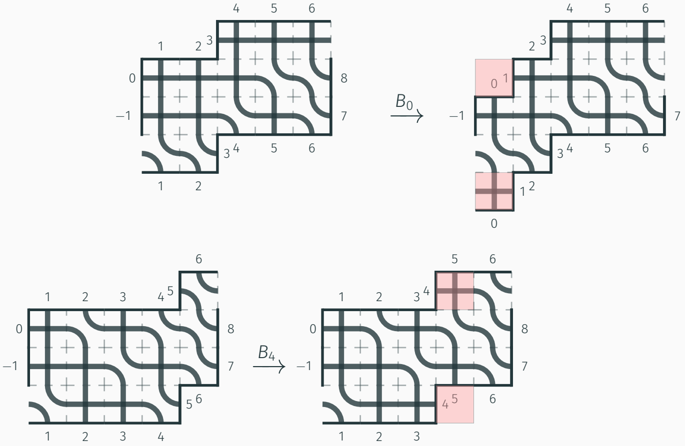

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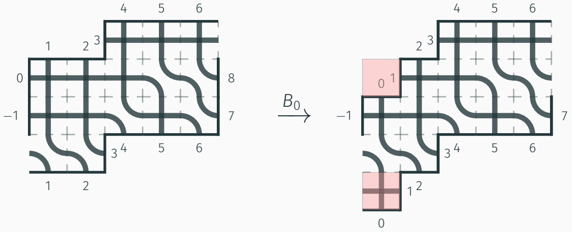

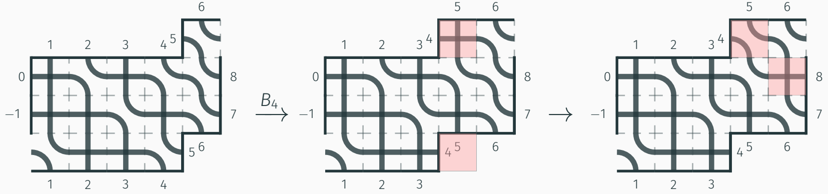

The move B_0 is why we need affine Deograms. It has no simple "lift" to rectangular Deograms.

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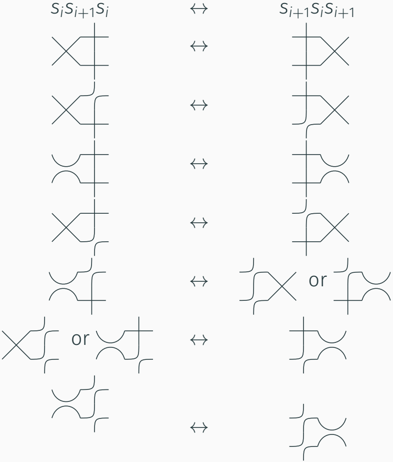
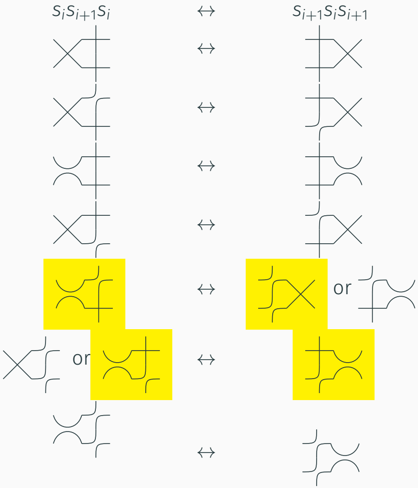
Yang-Baxter Moves



No bijection...

Yang-Baxter Moves

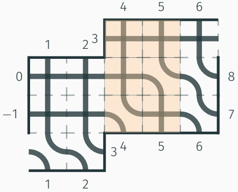


No bijection...

Bijection if we require Condition 3. (No elbow after an odd number of crossings)

Zipper

Motto: We cross wires below and locally apply Yang-Baxter moves until the crossing moves to the top of the path.



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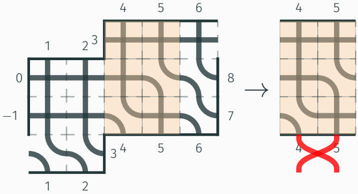

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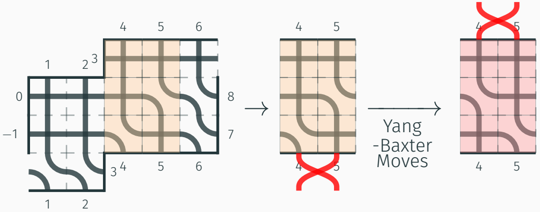

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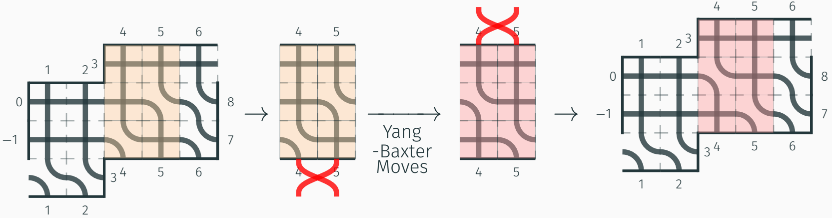

Moves on Affine Deograms

We have 3 moves on f-affine Deograms:

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Decoupling

Let $f = f_1 f_2 \dots f_r$ be a decomposition of $f \in \mathsf{B}_{k,n}$ into cycles. Then,

$$\#\,\mathsf{AffDeo}^{\mathsf{max}}_{f,P} = \prod_{i=1}^{r} \#\,\mathsf{AffDeo}^{\mathsf{max}}_{f_i,P_i}.$$

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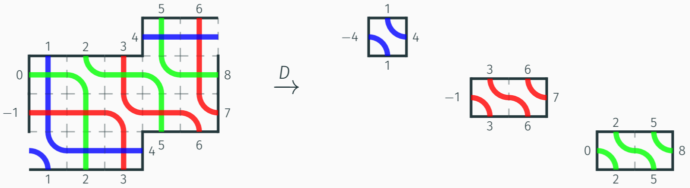
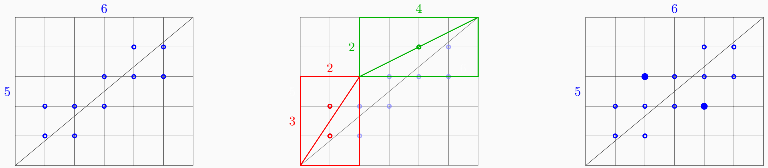


Dyck Path and (Affine) Deogram Recurrence

Let f_1, f_2 the cycles obtained by resolving f at $i, i+1$, and $f' = s_i f s_i$.



$$\mathrm{Dyck}(\Gamma(f)) = \mathrm{Dyck}(\Gamma(f_1)) \cdot \mathrm{Dyck}(\Gamma(f_2)) + \mathrm{Dyck}(\Gamma(f'))$$

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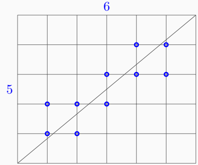

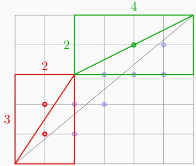

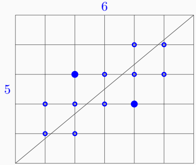

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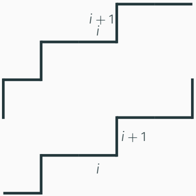

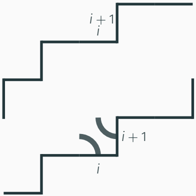

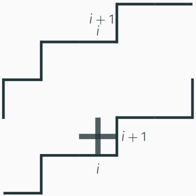

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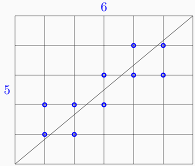

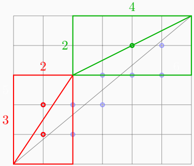

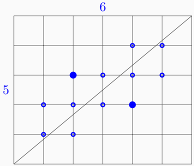

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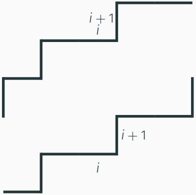

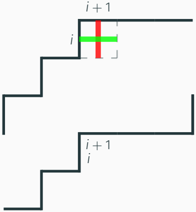

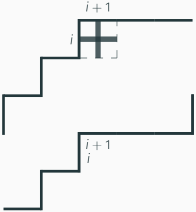

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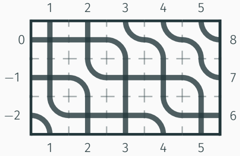
Full Recurrence Example



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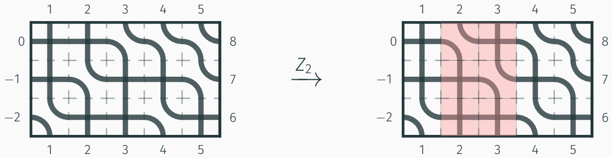

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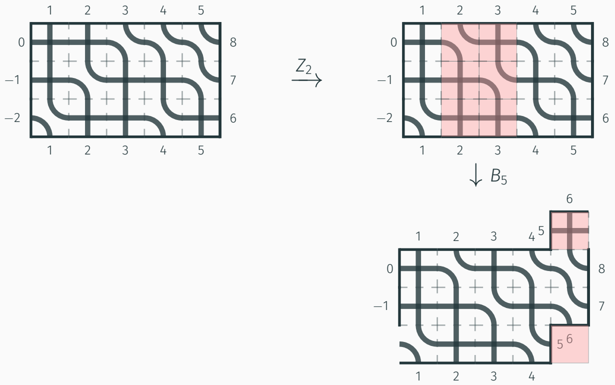

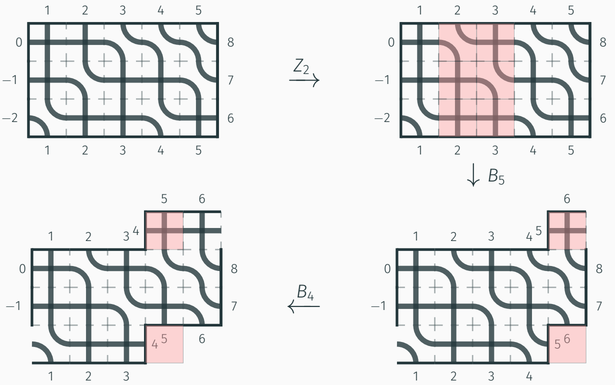

1 2 3 4 5
0 −1 −2
8 7 6
1 2 3 4 5
Z_2
1 2 3 4 5
0 −1 −2
8 7 6
1 2 3 4 5
B_5
5 6
1 2 3 4
0 −1
8 7
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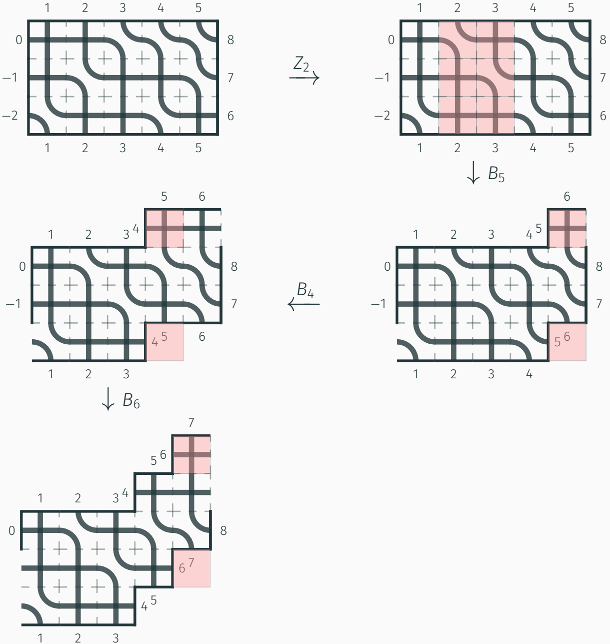

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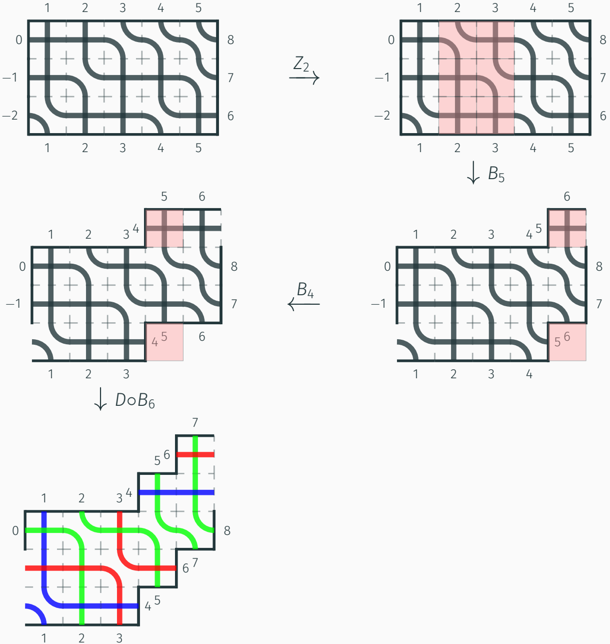

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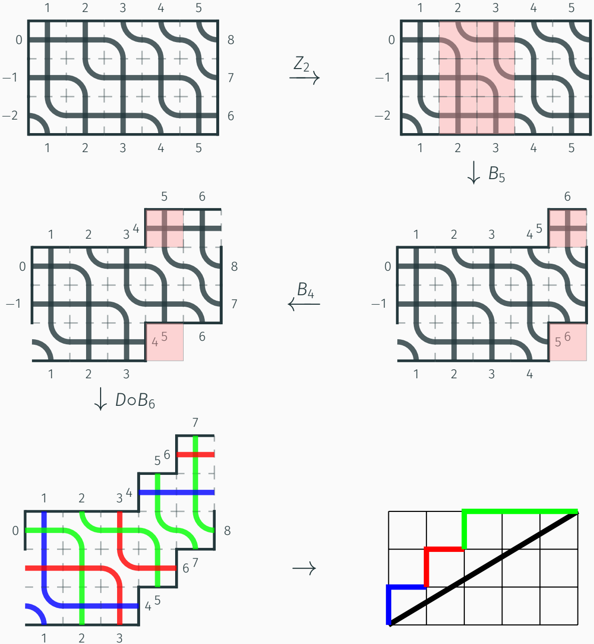

Possible directions

Question
Can we make this bijection direct?

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So far, yes for:

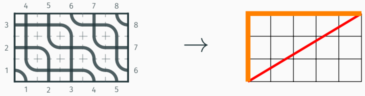
1. Catalan case, i.e., $(k, k+1)$. (Galashin Lam, '23)
2. 2-row and 2-column case. (M., '25+)

Possible directions

Dyck paths carry a lot of statistics.

$$C_{k,n}(q,t) = \sum_{D \in \mathrm{Dyck}_{k,n}} q^{\mathrm{area}(D)} t^{\mathrm{dinv}(D)}.$$

Question
Can we find statistics on Deograms which makes the bijection statistic-preserving? Can we bijectively prove these statistics are symmetric?



Questions?

Geometric Background

For every $f \in \mathsf{B}_{k,n}$, let $C_f = \chi_T(\Pi_f^\circ)$, the toric-equivariant Euler characteristic of the positroid variety associated to f. Then $C_f = \# \mathsf{AffDeo}_{f,P}$, when P is the first element of the Grassmannian necklace for f.

This is also related to

1. Kazhdan-Lusztig R-polynomials,
2. HOMFLY polynomials,
3. Khovanov-Rozansky triply-graded link invariants.