The Combinatorics of Affine Deodhar Diagrams

SUNY Binghamton Combinatorics Seminar

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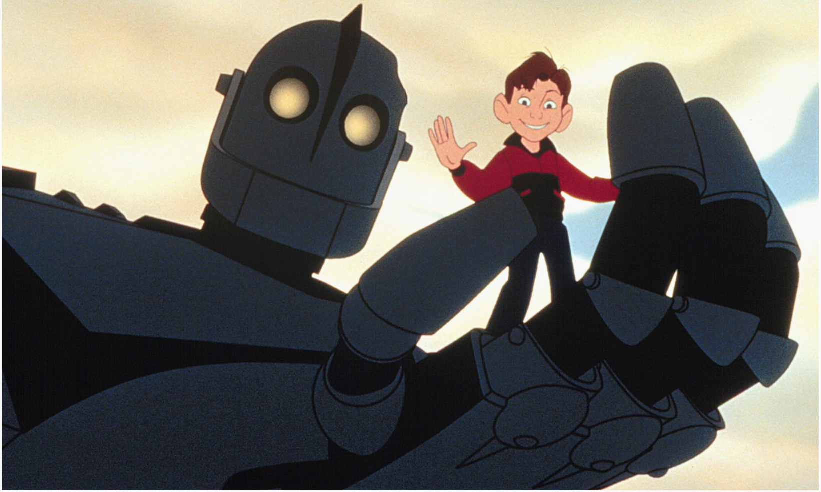
UC Los Angeles

Combinatorics vs Geometry?

Combinatorics and Geometry?

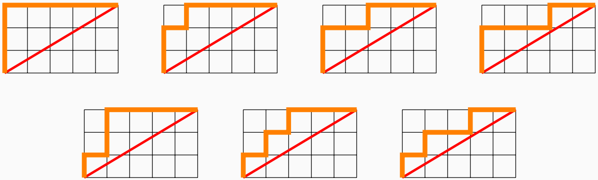


Outline

1. Rational Catalan Combinatorics
2. Definition of Positroids and (Affine Patches) of Positroid Varieties,
3. Definition of Affine Deograms
4. Combinatorial Bijections vs/and Geometric Isomorphisms of Affine Deograms

Rational Catalan Numbers

For any $0 < k < n$ with $\gcd(k, n) = 1$, we define the **rational Catalan number** $C_{k,n-k}$ as the number of Dyck paths in a $k \times (n-k)$ rectangle.



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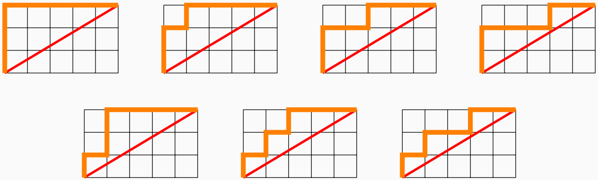


Closed Formula for $C_{k,n-k}$.

For any $0 < k < n$ with $\gcd(k, n) = 1$,

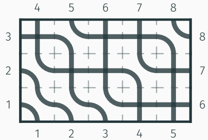
$$C_{k,n-k} = \frac{1}{n}\binom{n}{k}.$$

When $n = 2k + 1$, we recover the classical Catalan numbers.

(k, n)-Deograms

(k, n)-Deodhar Diagrams

A (k, n)-Deodhar Diagram (**Deogram**) is a filling of boxes of a $k \times (n-k)$ rectangle with crossings, , and elbows, , with



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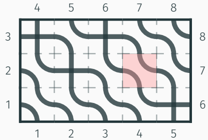

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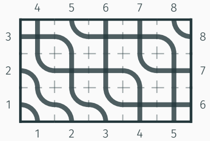

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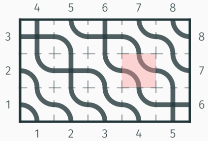

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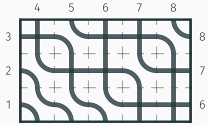

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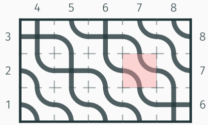

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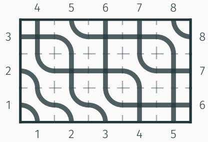

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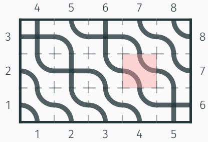

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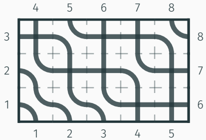

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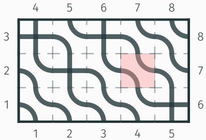


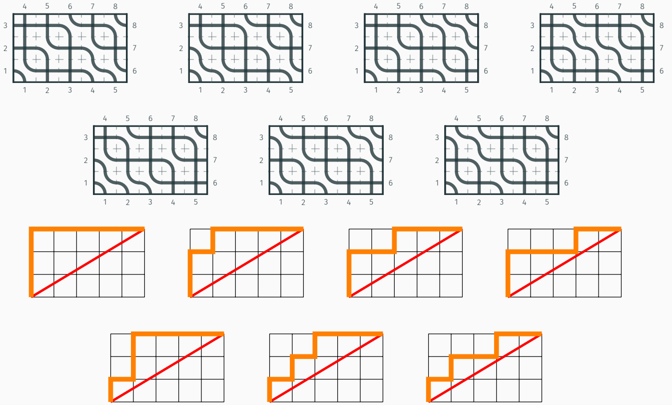
Non-example

Let $\mathrm{Deo}_{k,n}$ denote the set of (k, n)-Deograms.

Combinatorial Overview

Theorem (Galashin-Lam, '21)

For $0 < k < n$ with $\gcd(k, n) = 1$, $\mathrm{Deo}_{k,n}$ and $\mathrm{Dyck}_{k,n}$ are equinumerous.



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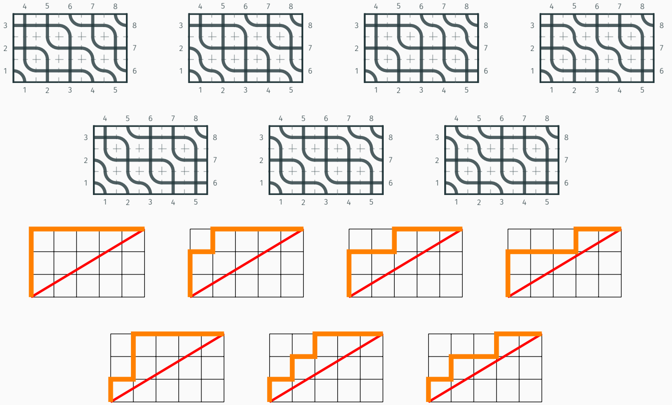

No general bijective proof!

Plücker Coordinates

Let $0 < k \leq n$. Define the **Grassmannian**

$$\begin{aligned} \mathrm{Gr}(k,n) &= \{V \subseteq \mathbb{C}^n \mid \dim(V) = k\} \\ &= \{\text{full rank } k \times n \text{ matrices}\}/\{\text{row operations}\} \end{aligned}$$

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Row operations change the Plücker coordinates by a common rescaling. This gives rise to the Plücker embedding $\mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$.

Positroids: Two Definitions

A matroid $\mathcal{M}$ on $[n] := \{1, 2, \ldots, n\}$ of rank k is a non-empty collection of k-element subsets of $[n]$, called bases satisfying:

For any permutation $w \in S_n$, there is a unique minimal element of $w \cdot \mathcal{M}$ in the partial order $\leq$ on $\binom{[n]}{k}$.

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$M(A)$ is called a **positroid** if $\Delta_I(A) \geq 0$ for all $I \in \binom{[n]}{k}$.

Positroid Example

Revisiting our last example.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

$$\Delta_{12}(A) = 1$$
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Observation

$\{\mathbf{a_i} \mid i \in I\}$ form a basis for $\mathbb{F}^k \iff \Delta_I \neq 0$.

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So, we get the positroid $M(A) = \{12, 23, 13\}$.

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First extend A to a $k \times \mathbb{Z}$ matrix by repeating A.

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The set $M(A) = \{I \in \binom{[n]}{k} \mid I_k \leq_k I \text{ for all } k \in [n]\}$ is the positroid of A.

Positroid Example

Now a larger example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{array}{ll} \Delta_{123}(A) = 1 & \Delta_{145}(A) = 1 \\ \Delta_{124}(A) = 1 & \Delta_{234}(A) = 0 \\ \Delta_{125}(A) = 1 & \Delta_{235}(A) = 1 \\ \Delta_{134}(A) = -1 & \Delta_{245}(A) = 1 \\ \Delta_{135}(A) = 0 & \Delta_{345}(A) = -1 \end{array}$$

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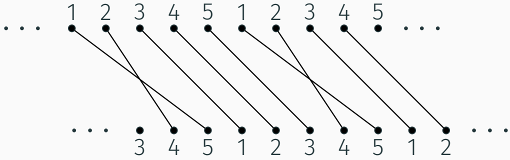

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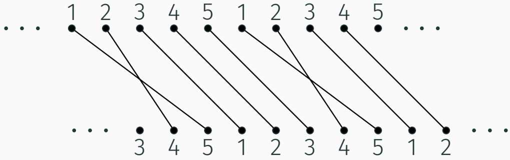

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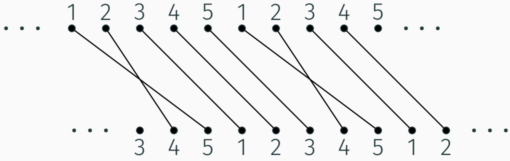

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We get the positroid $M(A) = \binom{[5]}{3} \setminus \{234\}$.

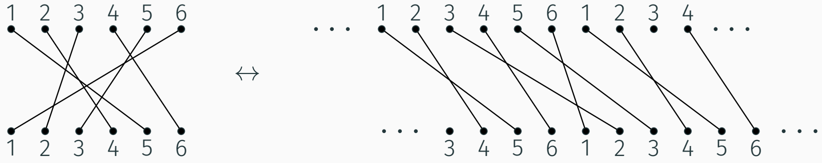
What permutations arise via this process?

Bounded Affine Permutations

What permutations arise via this process?

A bijection $f : \mathbb{Z} \to \mathbb{Z}$ is a (k,n)-**bounded affine permutation** if:

1. $f(i+n) = f(i) + n$ for all $i \in \mathbb{Z}$,
2. $i \leq f(i) \leq i + n$ for all $i \in \mathbb{Z}$,
3. $\sum_{i=1}^{n}(f(i) - i) = kn$.



We let $\mathsf{B}_{k,n}$ denote the set of (k,n)-bounded affine permutations.

Positroid

For any $f \in \mathsf{B}_{k,n}$, we define the open positroid variety

$$\Pi_f^\circ = \{X \in \mathrm{Gr}(k,n) \mid f_X = f\}.$$

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Theorem (Knutson-Lam-Speyer, 2013)

There exists a stratification of the Grassmannian

$$\mathrm{Gr}(k,n) = \bigsqcup_{f \in \mathsf{B}_{k,n}} \Pi_f^\circ.$$

Decomposing positroid varieties

To better understand positroid varieties, we decompose them into simpler pieces.

Theorem (Deodhar, 1985)

$$\Pi_f^\circ \cong \bigsqcup_{D \in \mathrm{Deo}_f} (\mathbb{C}^*)^{\#elbows(D)} \times \mathbb{C}^{(\#xing(D) - \ell(f))/2}$$

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Corollary

Let q be a prime power. Then,

$$\Pi_f^\circ(\mathbb{F}_q) \cong \bigsqcup_{D \in \mathbf{Deo}_f} (\mathbb{F}_q^*)^{\#elbows(D)} \times \mathbb{F}_q^{(\#xing(D) - \ell(f))/2}.$$

So,

$$\#\Pi_f^\circ(\mathbb{F}_q) = \sum_{D \in \mathbf{Deo}_f} (q-1)^{\#elbows(D)} q^{((\#xing(D) - \ell(f))/2)} = R_f(q).$$

Motivating Question

Motivation (Affine Patches of Positroid Varieties)

What if I wanted to study pieces of the form

$$\Pi_f^\circ \cap \{\Delta_I \neq 0\}?$$

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Why?

It is a Theorem of Snider (2010) that there exists a birational isomorphism

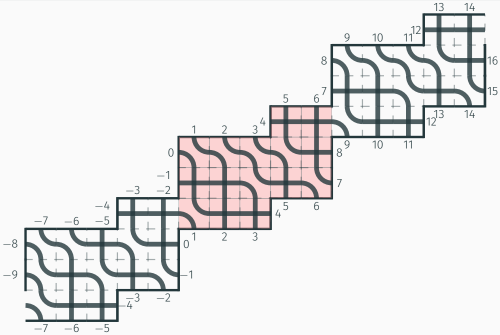
$$\Pi_f^\circ \cap \{\Delta_I \neq 0\} \cong \mathring{\mathcal{R}}_f^{g_I},$$

where $\mathring{\mathcal{R}}_f^{g_I}$ is an (open) **affine Richardson variety** in the **affine flag variety**.

Main Tool: Affine Deograms

Definition (M., 26+)

An (f, P)-**affine Deogram** is a *periodic filling* of the space between a path P with k up-steps and $n-k$ right steps and its vertical translate such that:



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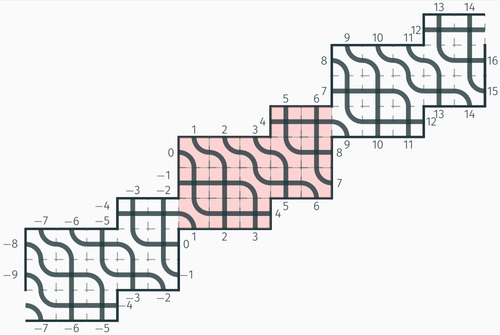

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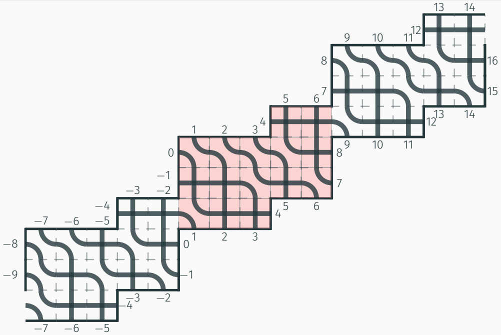

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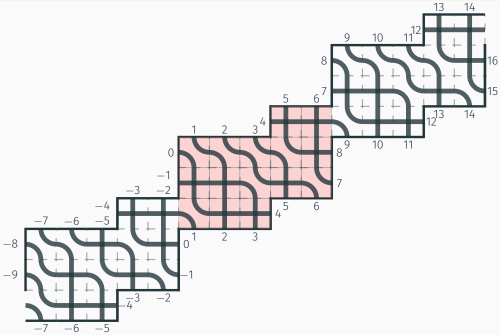

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Bijection with (k, n)-Deograms

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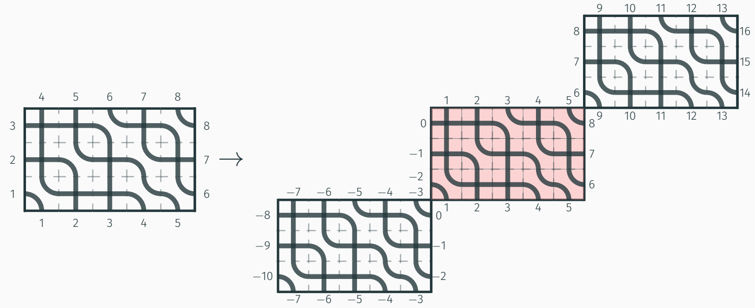
We have a bijection $\mathrm{Deo}_{k,n} \to \mathrm{AffDeo}_{f_{k,n},P_{k,n}}$.



Affine Deograms, Geometrically

We get a formula for the point count! Specifically,

$$\#(\Pi_f^\circ \cap \{\Delta_{P_U} \neq 0\})(\mathbb{F}_q) = \sum_{A \in \mathsf{AffDeo}_{f,P}} (q-1)^{\#\mathrm{elbows}(A)} q^{((\#\mathrm{xing}(A) - \ell(f))/2)}.$$

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Box Addition/Removal + Zipper

Suppose $s_i f > f$. We have isomorphisms

$$\Pi_f^\circ \cong \begin{cases} \Pi_{s_i f s_i}^\circ, & fs_i < f; \\ (\Pi_{s_i f s_i}^\circ \times \mathbb{C}) \sqcup (\Pi_{fs_i}^\circ \times \mathbb{C}^*), & fs_i > f. \end{cases}$$

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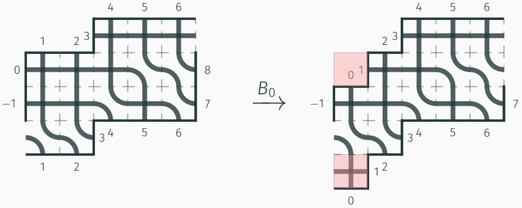
$$\mathrm{AffDeo}_{f,P} \cong \begin{cases} \mathrm{AffDeo}_{s_i f s_i, s_i P}, & f s_i < f; \\ \mathrm{AffDeo}_{f s_i, s_i P} \sqcup \mathrm{AffDeo}_{s_i f s_i, s_i P}, & f s_i > f. \end{cases}$$

If $s_i P \neq P$, this is *Box Addition/Removal.*

If $s_i P = P$, this is *Zipper.*

Box Addition/Removal

Motto: We change our path at index i and move the box up/down



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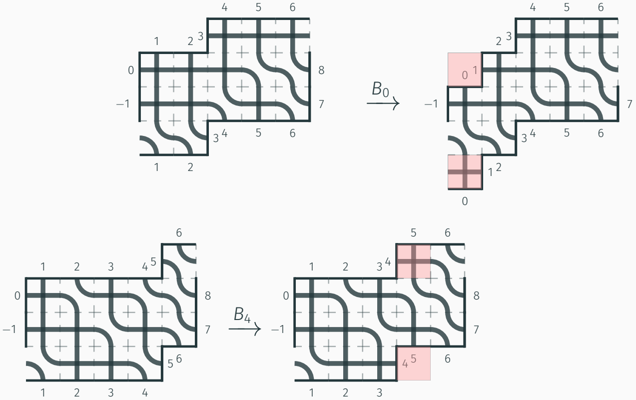

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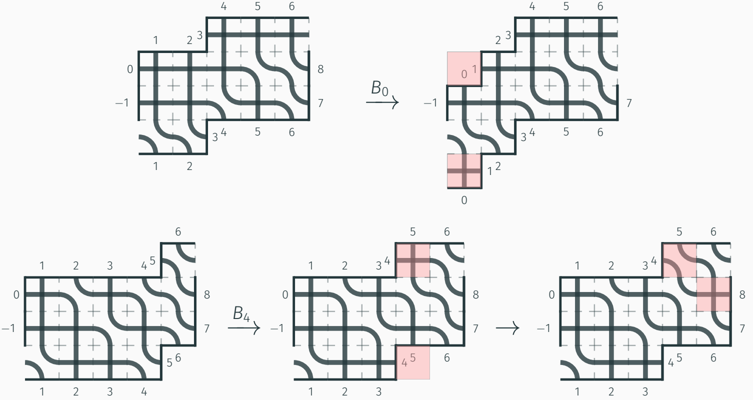

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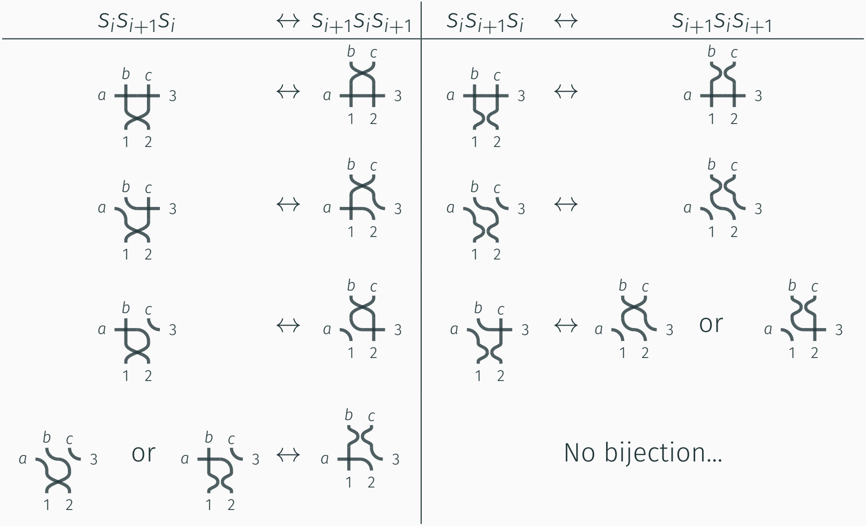
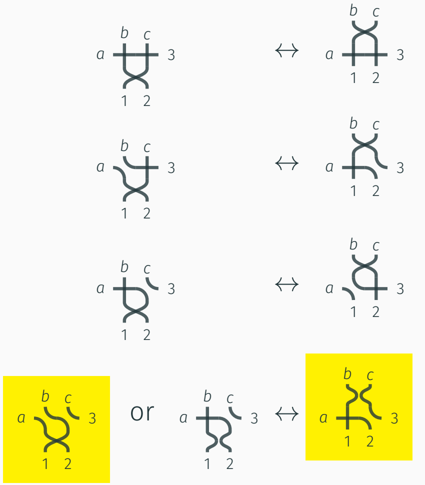
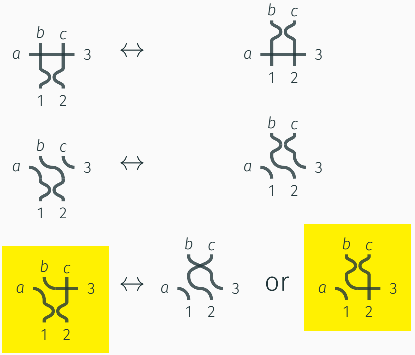
Yang-Baxter Moves



Yang-Baxter Moves

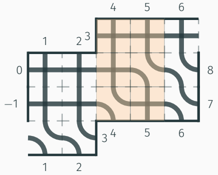
$s_is_{i+1}s_i$ $\leftrightarrow$ $s_{i+1}s_is_{i+1}$	$s_is_{i+1}s_i$ $\leftrightarrow$ $s_{i+1}s_is_{i+1}$





Bijection if we require the distinguished condition!

Motto: We cross wires below and locally apply Yang-Baxter moves until the crossing moves to the top of the path.



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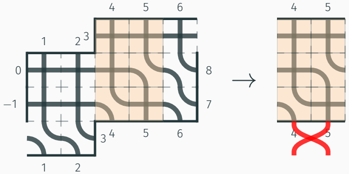

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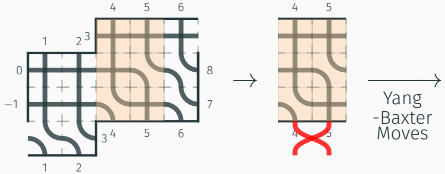

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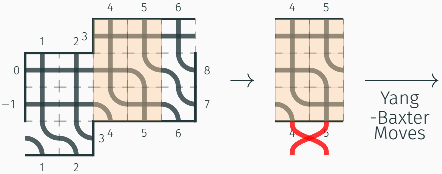

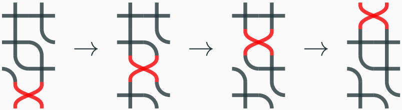

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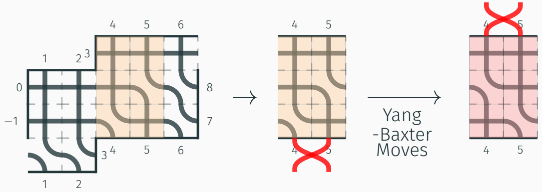

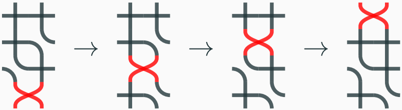

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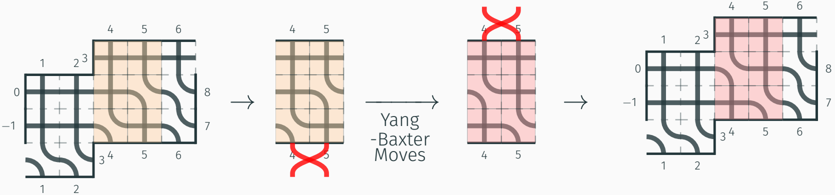

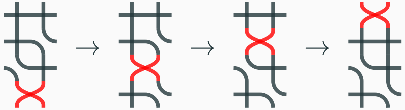

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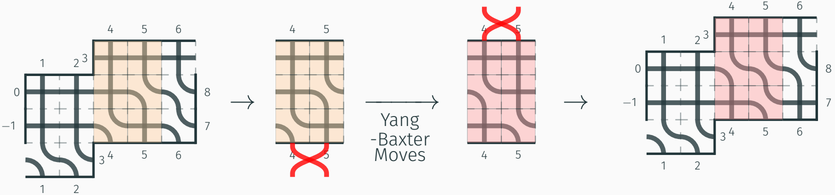

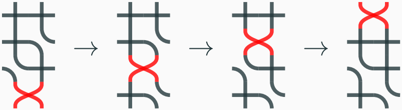

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Decoupling

Let $f = f_1 f_2 \dots f_r$ be a cycle decomposition of $f \in \mathsf{B}_{k,n}$. Then,

$$\# \operatorname{AffDeo}^{\max}_{f,P} = \prod_{i=1}^{r} \# \operatorname{AffDeo}^{\max}_{f_i,P_i}.$$

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Corollary

Let $f = f_1 f_2 \dots f_r$ be a cycle decomposition of $f \in \mathsf{B}_{k,n}$. Then,

$$\chi(\Pi^\circ_f/T) = \prod_{i=1}^{r} \chi(\Pi^\circ_{f_i}/T).$$

Geometric proof..?

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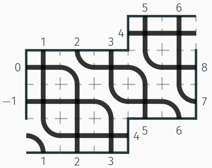

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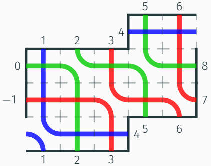

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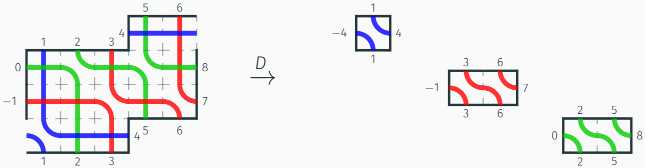

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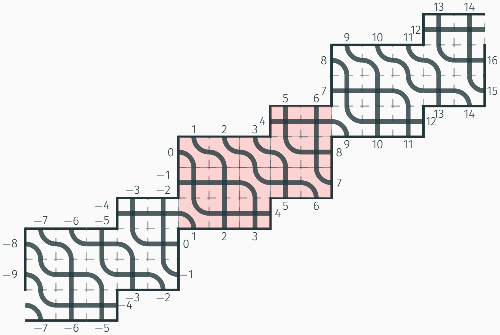

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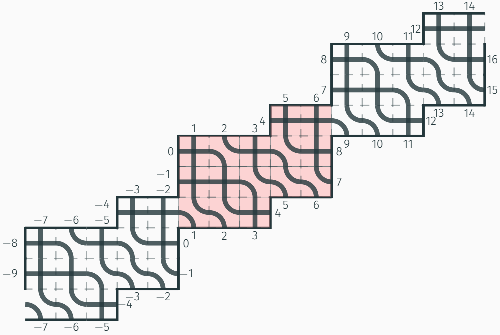
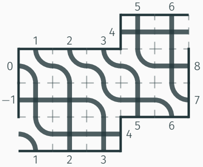


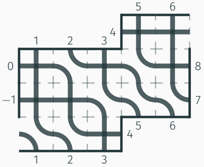
Distinguished vs. Anti-Distinguished

No elbows after an odd number of crossings

Distinguished (from the top)



Anti-Distinguished (from the bottom)



Twist Bijection

Let $\mathbf{f} = s_{i_1} \dots s_{i_r}$ be a reduced word for f. We have a bijection

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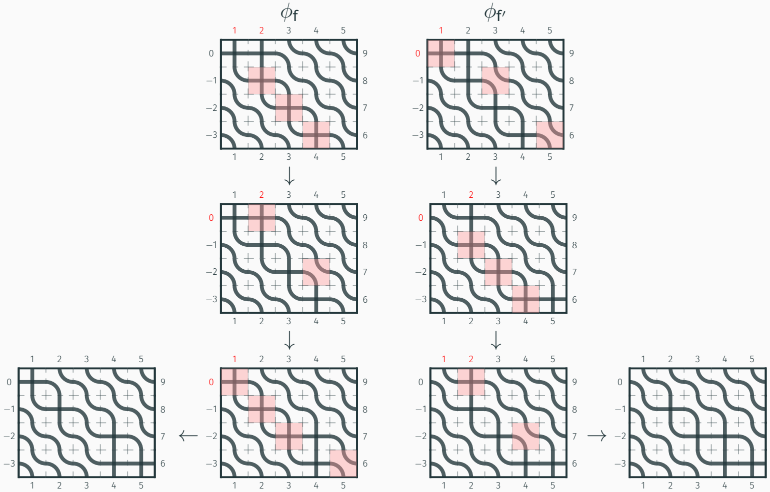
Let $\mathbf{v} = s_{i_1} \dots s_{i_r}$ be a reduced word for v and $v \leq w \in S_n$. We have a chain of isomorphisms of *twisted* Richardson varieties.

$$R_{v,w} \to R_{v,w}^{(s_{i_r})} \to R_{v,w}^{(s_{i_r} s_{i_{r-1}})} \to \dots \to R_{v,w}^{(v^{-1})}.$$

This allows us to turn *leftmost* subexpressions to *rightmost* subexpressions.

Two Examples of the Twist Bijection



Geometry Informing Combinatorics

There exist some geometric isomorphisms which imply two sets being equinumerous. We do not have bijections yet for these cases.

1. Cyclic Symmetry
2. Weak Separation

Cyclic Symmetry

Let σf denote the cyclic shift of i. That is, $(\sigma f)(i) = f(i+1) - 1$. Then,

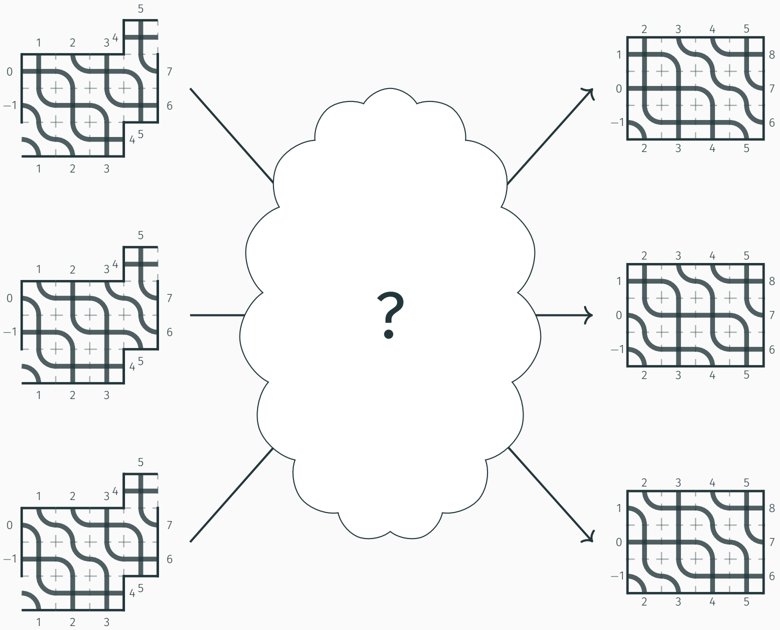
$$\Pi_f^\circ \cong \Pi_{\sigma f}^\circ.$$

This implies the equality

$$\#\,\mathrm{AffDeo}_{f,P_{I_1(f)}} = \#\,\mathrm{AffDeo}_{f,P_{I_2(f)}},$$

where I_1, I_2 are the first and second Grassmann necklace elements of f. However we do not have a combinatorial proof of this fact.

Cyclic Symmetry



Weak Separation

Positroid varieties may be parametrized by plabic graphs. More explicitly, the coordinate ring $\mathbb{C}[\Pi_f^\circ]$ is a cluster algebra whose cluster variables are face labels of a reduced plabic graph*.

Weak Separation [Oh-Postnikov-Speyer, 2011]

I appears as a face label of a plabic graph if and only if I is weakly separated to $\mathcal{I}(f)$, the Grassmann necklace of f.

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This implies that the equality, when $\ell(fs_i) = \ell(s_i f)$

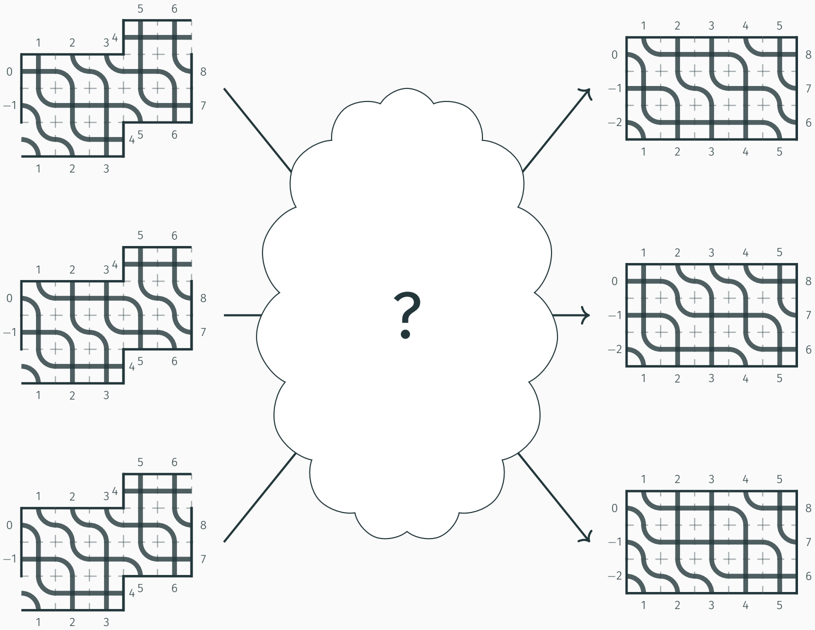
$$\#\operatorname{AffDeo}_{s_i f, I} = \#\operatorname{AffDeo}_{fs_i, J},$$

where

$$J = \begin{cases} s_i I, & I \neq I_{i+1}(f), \\ t_{f^{-1}(i), f^{-1}(i+1)} I_{i+1}, & I = I_{i+1}(f). \end{cases}$$

However we do not have a combinatorial proof of this fact for the final case.

Weak Separation



Bijecting Maximal Deograms with Rational Dyck Paths

Problem 1

For any $f \in \mathsf{B}_{k,n}$ and $i \in [n]$, find a bijection

$$\mathrm{AffDeo}_{f,P_{I_i(f)}} \to \mathrm{AffDeo}_{f,P_{I_{i+1}(f)}}.$$

Problem 2

For any $fs_i, s_if \in \mathsf{B}_{k,n}$ with $\ell(fs_i) = \ell(s_if)$, find a bijection

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These are harder than the aforementioned bijections, as our shapes undergo a *ribbon change*, not just a *box change*.

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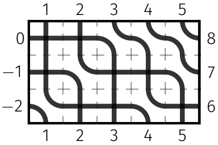
Solving either Problem 1 or Problem 2 will provide a combinatorial bijection $\mathsf{Deo}_{k,n} \to \mathsf{Dyck}_{k,n}$.

Thank you!



$\xrightarrow{?}$